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Right projective semigroups with 0 ☆

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Abstract

We describe the semigroups S with 0 that are projective in the category of right S-objects, i.e. of centered right S-operands. Among them are the free semigroups with an adjoined zero, and several characterizations of this subclass are given. Two other classes, one of them close to free semigroups with 0, the other rather far apart, are discussed in detail. The first class consists of semigroups of finite and infinite sequences, with multiplication based on concatenation. The second is formed by the projectives in the category of (S,S)-biobjects. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Every monoid K is free as a unital right K-operand ($=K$-set$=K$-act$=K$-action $=$ unital K-system), i.e. K_K is free, hence projective. If K has a zero element then K_K belongs to the subcategory of centered unital right K-operands and K_K is projective within this subcategory. If we consider semigroups S and decide to drop the unitality condition then not every S turns out to be free as a right operand, and not even projective. Moreover, if S has a zero element then it matters whether or not we consider S_S within the subcategory of centered right S-operands or within the full category. Certainly, if S is a semigroup with 0, i.e. if we consider the zero element of S as part of the underlying structure (in the same sense in which an additive group underlies a ring), it is natural to admit only objects with a distinguished invariant element.

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More importantly, we obtain somewhat more general results if we choose this smaller category. An object of this category will be called a right S-object.

The following is the central observation. Let S be a semigroup with 0, let I denote the intersection of the ideals S^n, $n=1,2,3,\dots$ and B the intersection of the right ideals $(S\backslash S^2)^n I \cup \{0\}$, $n=1,2,3,\dots$. Then S is projective as a right S-object iff (1) every maximal indecomposable right ideal R is a principal right ideal, and isomorphic with an idempotent-generated right ideal in the case $R=RS$, (2) $S\backslash S^2$ is a set of left cancellable elements and (3) $aI \cap bI = \{0\}$ for all $a,b \in S\backslash S^2$ such that $a \neq b$. In any such semigroup S there is a largest subsemigroup S' whose maximal indecomposable right ideals of the form $R'=R'S'$ coincide with the maximal indecomposable right ideals of S that are isomorphic with idempotent-generated right ideals contained in B. Moreover, $S' \cap I$ is a right ideal of S and S is a 0-disjoint union of S' with another subsemigroup S'', which is a right ideal of S and satisfies $S''^2 = S''$. Both S' and S'' are semigroups with 0 that are projective as right objects over themselves. We concentrate on the structure of S' and on the composition of S' and S'' (Section 3), leaving a more detailed description of the semigroups S'' to another paper. An exception is the characterization of the semigroups S which are projective in the category of left-right biobjects over S (Section 6), a very special class of semigroups of the form $S=S''$ which can be described in terms of Rees matrix semigroups.

In an arbitrary right projective semigroup with 0 the set $S\backslash S^2$ generates a free subsemigroup. If this subsemigroup is non-cyclic then I is either the zero ideal or non-finitely generated as a right ideal. From this we deduce that S is a free semigroup with 0 iff S is projective as a right S-object, I is finitely generated as a right ideal, $S^2 \neq S$, and every non-zero element of S is cancellable. There are several further characterizations of free semigroups (Section 4). A large class of semigroups S such that S_S is a free S-object is obtained by adjoining 0 to the set of sequences over a set A, of length smaller than some infinite ordinal β, and extending concatenation to an associative operation. The simplest case beyond $\beta=\omega$ (the free semigroups with 0) is $\beta=\omega+1$. These semigroups can be classified to some extent (Section 5). Generalizations of different sorts are discussed in the final section.

2. Preliminaries

Let S be a semigroup with 0. By a right S-object we understand a set M with an associative scalar multiplication $(m,s) \mapsto ms$ from $M \times S$ to M together with a distinguished element $0_M \in M$ such that $m0 = 0_M$ for all $m \in M$. Such structures have been called centered operands over S in [6]. Obviously, S can be regarded as a right S-object. The element 0_M of any right S-object M will always be denoted by 0 and called the zero element of M. Is is often feasible to assume that the zero elements of all right objects in question coincide with the zero element of S and this we contend when we speak of a 0-disjoint union.

The category of right S-objects is formed by choosing for every pair (M,N) of right S-objects the full set of homomorphisms, i.e. of mappings $\phi : M \mapsto N$ such that $\phi(ms) = \phi(m)s$ for all $s \in S$ and all $m \in M$. Clearly, any such mapping satisfies $\phi(0) = 0$. The notation M_S indicates that we consider M as an object of this category. Especially, this holds if we write S_S.

As in any category [12], M is called projective if every homomorphism $h : M \mapsto N$ factors through any epimorphism $g : B \mapsto N$ (i.e. $h = g \circ h'$ for some homomorphism $h' : M \mapsto B$). It is easily seen that the epimorphisms between right S-objects coincide with the surjective homomorphisms. A right S-objects M is called free (over the set X) if there exists a subset Y of M such that every mapping $f : Y \mapsto N$ into a right S-object N can be uniquely extended to a homorphism $h : M \mapsto N$ (and $X = Y$) [1, 8.22]. We shall say that S is right projective or right free if S_S is projective or free, respectively.

Let $S^{\underline{1}}$ be the monoid with 0 obtained from S by adjunction of a new element as identity element. The category of unital right $S^{\underline{1}}$-objects is isomorphic with the category of right objects over S under the trivial forgetful functor. We will not distinguish between these categories (so that we can write $m1$ instead of m for any $m \in M$). It is easily seen that $S^{\underline{1}}_S$ is free over $\{1\}$ (the one-element set consisting of the adjoined element).

The following connection between free and projective right S-objects is known to hold under very general circumstances [1, 9.29]. Let M be projective and $g : F \mapsto M$ an epimorphism. Then M is a retract of F, i.e. isomorphic with the image of F under an idempotent endomorphism (take h to be the identity homomorphism, then any h' with $g \circ h' = h$ is a suitable isomorphism and $h' \circ g$ a suitable endomorphism). Since M is the image of a free right S-object (consider a free right S-object over the set M—we shall see that such a right S-object exists—and let $f : M \mapsto M$ be the identity mapping), we have that any projective right S-object is the retract of a free one. Conversely, a retract of a projective right S-object is easily seen to be projective. Moreover, any free right S-object is projective (if F is free over X then, for $h : F \mapsto N$ and epimorphism $g : B \mapsto N$, choose a mapping f from X into B such that $g(f(x)) = h(x)$ and let h' be the homomorphic extension of f). Hence any retract of a free right S-object is projective, i.e. the retracts of free ones and the projectives coincide.

The following observations are straightforward. Let M be an S-object (briefly for "right S-object"). By an S-subobject we understand a subset U such that $US \subseteq U$ (together with 0 and the obvious restriction of the scalar multiplication). Clearly, the subobjects of S_S are the right ideals. M is a coproduct of a family $(N_\iota)_{\iota \in I}$ of S-objects [1, 12] iff M is a 0-disjoint union of subobjects U_ι (i.e. $M = \bigcup_{\iota \in I} U_\iota$ and $U_\iota \cap U_\kappa = 0$ for all $\iota \neq \kappa$) where each U_ι is isomorphic with N_ι. Clearly, M is a coproduct of the family $(U_\iota)_{\iota \in I}$. We reserve the notation $M = \coprod_{\iota \in I} U_\iota$ for this special situation and we call a subobject N a cofactor if N belongs to any such family $(U_\iota)_{\iota \in I}$. Every cofactor is a retract of M (but not conversely)—the zero element is essential here. Observe that we let $\bigcup$ denote the join in the lattice of subsets containing 0 and that the 0-disjoint union of the empty set of subobjects is defined to be $\{0\}$ (so that $\{0\}$ is the coproduct of the empty family). However, we use $\cap$ and $\cup$ in the ordinary set-theoretic sense.

A non-zero S-object N is called indecomposable (as a coproduct) if, for non-zero subobjects U_1 and U_2, $N = U_1 \cup U_2$ implies $U_1 \cap U_2 \neq \{0\}$ (considered as an S-operand such an N has been called 0-indecomposable in [6]). Clearly, every non-zero element m of an S-object M is contained in an indecomposable subobject, e.g. $mS^{\underline{1}}$ is indecomposable. If U and V are indecomposable subobjects such that $U \cap V \neq \{0\}$ then $U \cup V$ is easily seen to be indecomposable. This argument may be used to show that any indecomposable subobject U is contained in a maximal indecomposable subobject. Indeed, let $\hat{U}$ be the union of all indecomposable subobjects of M with a non-zero intersection with U. Then $\hat{U}$ is indecomposable and any indecomposable subobject with a non-zero intersection with $\hat{U}$ is contained in $\hat{U}$. Thus $\hat{U}$ is a maximal indecomposable subobject. Moreover, M is a 0-disjoint union, hence a coproduct, of its maximal indecomposable subobjects.

Let $\mathscr{M}(M_S)$ be the set of maximal indecomposable right S-subobjects of M, and let $Q(M_S)$ the join of all $U \in \mathscr{M}(M_S)$ such that $US \subset U$, and $P(M_S)$ the join of all $U \in \mathscr{M}(M_S)$ such that $US = U$. Thus M is the 0-disjoint union of $Q(M_S)$ and $P(M_S)$. Let X be a subset of M. If M is free over X then, for any $U \in \mathscr{M}(M_S)$, U is free over $U \cap X$ (map every $m \in X \setminus U$ to 0). Conversely, if every $U \in \mathscr{M}(M_S)$ is free over $U \cap X$, then M is free over X (by the coproduct property). The free S-object $S_S^{\underline{1}}$ is indecomposable, which implies that any free S-object M is isomorphic with a coproduct of S-objects isomorphic with $S_S^{\underline{1}}$ (observe that free objects over sets of equal cardinality are isomorphic). Obviously, in this case $M = Q(M_S)$. We also see that free S-objects over any set S exist (by choosing for every $x \in X$ an S-object isomorphic with $S_S^{\underline{1}}$ and building the coproduct).

Now consider an indecomposable projective S-object V. We may assume (by isomorphism) that $V = f(F)$ for some retraction $f : F \mapsto F$ of a free S-object F. Then V is a subobject of an $U \in \mathscr{M}(F_S)$ and $V = f(U)$ (because $f(V) = V$). We may assume that $U = S_S^{\underline{1}}$. Then, putting $u = f(1)$, we have $u \cdot S^{\underline{1}} = V$. If $u = 1$ then V is free, otherwise $u = f(u) = f(1 \cdot u) = f(1) \cdot u = u^2$ and $U = f(S^{\underline{1}}) = f(1) \cdot S^{\underline{1}} = u \cdot S^{\underline{1}} = uS$, i.e. U is an idempotent-generated principal right ideal of S. Conversely, any idempotent-generated principal right ideal of S is (as a right S-object) a retract of $S_S^{\underline{1}}$. It is easily seen that a coproduct of projective objects is projective. Hence a right S-object M is projective iff M is a coproduct of right objects isomorphic with $S_S^{\underline{1}}$ or with idempotent-generated right ideals of S. In this case $P(M_S)$ is a coproduct of S-objects isomorphic with idempotent-generated principal right ideals of S, and $Q(M_S)$ is again a free S-object.

The analogues of the preceding paragraphs for a monoid K (not necessarily with a zero element) and K-sets M are standard matter [3,12,13], and may easily be translated into operands over a semigroup H. We briefly indicate—in order to justify a claim in the introduction—how the subsequent investigation relates to this situation.

A right operand M over H is projective in the category of all right H-operands iff $M^{\underline{0}}$, the operand over the semigroup $H^{\underline{0}}$ with 0, both being obtained by adjunction of a new zero element, is projective as an $H^{\underline{0}}$-object (as can be shown). If the semigroup H

has a zero element and is projective as a right H-operand then H is also projective as a right H-object (= centered H-operand, i.e. in the smaller category), but the converse does not hold. For if H is projective as an H-operand, then, writing S for $H^{\underline{0}}$, S is right projective, but has a smallest non-zero right ideal. The class of right projective semigroups with 0 having this form is quite restricted (Corollary 4.2). Put otherwise, there are more semigroups with a zero element that are projective as centered right operands than projective as general right operands (and the latter can even easily be identified among the former).

3. The components of right projective semigroups with 0

Let S be a semigroup with 0. We abbreviate $P(S_S)=\bigcup_{RS=R\in\mathscr{M}(S_S)} R$ by P and $Q(S_S)=\bigcup_{RS\subset R, R\in\mathscr{M}(S_S)} R$ by Q. Let A denote the set $S\backslash S^2$, $\langle A\rangle$ the subsemigroup generated by A, and put $T=\langle A\rangle\cup\{0\}$. Moreover, put $I=\bigcap_{1\leqslant i<\infty} S^i$ (an ideal) and $B=\bigcap_{1\leqslant i<\infty} T^i I$ (a right ideal). Clearly, A, T, I, and B all depend on S.

We have $P\subseteq I$, and it is easily seen that $A=Q\backslash QS$ and $S=\langle A\rangle\cup I$. In the case $A\neq\emptyset$ the former implies $AS\cap P=\{0\}$, hence $B\cap P=\{0\}$, and $A\cup AS\subseteq Q$. The supposition $A\neq\emptyset$ is needed to guarantee $0\in AS$. Moreover,

$$AB\subseteq B=\bigcap_{1\leqslant i<\infty} A^i I\subseteq\bigcap_{1\leqslant i<\infty} A^i S, \text{ and, with } A \text{ arbitrary,}$$

$$\{0\}\subseteq T\cap I,$$

$$\{0\}\subseteq B\cap TP,$$

and $B\cup\langle A\rangle P\subseteq I\cap Q$.

In general, all inclusions are proper, but we have the following assertions.

Lemma 3.1. *Let L be a non-zero left ideal of S, C a subset and $\langle C\rangle$ the subsemigroup generated by C. Assume that*

$$l_a: x\mapsto ax\ (x\in L) \text{ is injective for every } a\in C,$$

and

$$aL\cap bL=\{0\} \quad \text{if } a\neq b\ (a,b\in C).$$

Then $\langle C\rangle$ is a free semigroup over C or a cyclic semigroup and

$$C\cdot\bigcap_{1\leqslant i<\infty} C^i L=\bigcap_{1\leqslant i<\infty} C^i L.$$

Proof. If $a_1a_2\dots a_m=b_1b_2\dots b_n$ then $a_1a_2\dots a_mL=b_1b_2\dots b_nL$, hence $a_1L\cap b_1L\neq\{0\}$, so that $a_1=b_1$ and $a_2\dots a_mL=b_2\dots b_nL$. If $L=c_1c_2\dots c_kL$ then l_{c_1} is surjective, hence $aL\subseteq c_1L$ for all $a\in C$, and therefore $a=c_1$ for all $a\in C$. Therefore $\#C=1$, or

$a_1a_2\dots a_m = b_1b_2\dots b_n$ implies $m=n$ and $a_i=b_i$ for $i=1,2,\dots,n$, i.e. $\langle C\rangle$ is free over C.

Every $b\in\bigcap_{1\leqslant i<\infty} C^iL$ is of the form $b=a_1^{(1)}y_1=a_1^{(2)}a_2^{(2)}y_2=a_1^{(k)}a_2^{(k)}\dots a_k^{(k)}y_k=\cdots$ with $a_i^{(k)}\in C$ and $y_k\in L$ for all i,k. Then $a_1^{(1)}=a_1^{(2)}=\cdots=a_1^{(k)}=\cdots$ and $b'=y_1=a_2^{(2)}y_2=\cdots=a_2^{(k)}\dots a_k^{(k)}y_k=\cdots\in\bigcap_{1\leqslant i<\infty} C^iL$ and $b=ab'$ with $a=a_1^{(1)}$. □

Remark 3.2. If $\bigcap_{1\leqslant i<\infty} T^i=\{0\}$ then $\bigcap_{1\leqslant i<\infty} T^iS=B$.

Proof. Assume $x\in\bigcap_{1\leqslant i<\infty} T^iS\setminus\bigcap_{1\leqslant i<\infty} T^iI$. Then there exists a $k\geqslant 1$ such that, for every $i\geqslant k, x=a_1^{(i)}a_2^{(i)}\dots a_i^{(i)}y_i$ for elements $a_i^{(i)},\dots,a_i^{(i)}\in A$ and $y_i\in S\setminus I$, but then $y_i\in\langle A\rangle$ so that $x\in\langle A\rangle^i$ which contradicts $\bigcap_{1\leqslant i<\infty}\langle A\rangle^i\cup\{0\}=\bigcap_{1\leqslant i<\infty} T^i=\{0\}$. □

The statement applies if $\langle A\rangle$ is a free semigroup.

Proposition 3.3. *Let S be such that*
(a) *every $a\in A$ is left cancellable,*
(b) $aI\cap bI=\{0\}$ *for all $a,b\in A$ such that $a\neq b$, and*
(c) $A\cup AS=Q$.
Then $\langle A\rangle$ is a free semigroup, $\langle A\rangle\cap I=\emptyset$, $TB=B$, and $B\cap TP=\{0\}$.

Proof. The conclusions hold if $A=\emptyset$. If not then $\langle A\rangle$ is free and $TB=B$ by Lemma 3.1.

Assume $\langle A\rangle\cap I\neq\emptyset$. Then there exists an $x\in A^n\cap S^{n+1}$ for some n, $x=a_1a_2\dots a_n=s_1s_2\dots s_{n+1}$. Suppose that n is minimally chosen. By definition of A, $n\geqslant 2$. Then x is left cancellable, hence $\neq 0$ (because $\langle A\rangle$ is free by 3.1), and $s_1\in Q$ (because $x\in AS\subset Q$, hence $s_1\notin P$). Therefore $s_1\in\{b\}\cup bS$ for some $b\in A$. We may assume $s_1=b$ (because we can redefine s_2). Now $b=a_1$ would imply $a_2\dots a_n\in S^n$ which is impossible by the minimality of n. Therefore $b\neq a_1$, but $a_1a_2\dots a_nx=bs_2\dots s_{n+1}x$, and $a_2\dots a_nx, s_2\dots s_{n+1}x\in I$. Therefore $a_1a_2\dots a_na_1a_2\dots a_n=0$ which is impossible as $\langle A\rangle$ is free.

Now assume $A\neq\{0\}$ and $TP\cap B\neq\{0\}$, hence $\langle A\rangle P\cap\bigcap_{1\leqslant i<\infty} A^iS\neq\{0\}$ by Remark 3.2. Then there exists an $x\in A^nP\cap A^{n+1}S$ with $x\neq n$ for some n, $x=a_1a_2\dots a_n p=b_1b_2\dots b_{n+1}s$. Suppose that n is minimally chosen with $n\geqslant 1$. We have $b_2\dots b_{n+1}s\in I$ or $b_2\dots b_{n+1}s\in\langle A\rangle$. The latter is impossible because otherwise $x\in\langle A\rangle$ which contradicts $x\in I$ (because $P\subseteq I$). It follows that $a_1=b_1$, hence $a_2\dots a_np=b_2\dots b_{n+1}s$, and $n\geqslant 2$ (because $P\cap AS\subseteq P\cap Q=\{0\}$). This contradicts the minimality of n. □

Proposition 3.4. *Assume $Q=A\cup AS$. Then $Q=\langle A\rangle\cup B\cup\langle A\rangle P$.*

Proof. $\langle A\rangle\cup B\cup\langle A\rangle P\subseteq Q$ always holds. We have $S=\langle A\rangle\cup I=\langle A\rangle\cup(I\cap Q)\cup P$. Therefore $Q=A\cup AS=\langle A\rangle\cup A(I\cap Q)\cup AP$. We have $S=\langle A\rangle\cup(I\cap Q)\cup\langle A\rangle P\cup P$, trivially. Assume $S=\langle A\rangle\cup A^j(I\cap Q)\cup\langle A\rangle P\cup P$ for some $j\geqslant 0$. Then $S=\langle A\rangle\cup A^jQ\cup\langle A\rangle P\cup P$ $=\langle A\rangle\cup A^j\langle A\rangle\cup A^jA(I\cap Q)\cup A^jAP\cup\langle A\rangle P\cup P=\langle A\rangle\cup A^{j+1}(I\cap Q)\cup\langle A\rangle P\cup P$. Let x be an element outside of $\langle A\rangle \cup \langle A\rangle P \cup P$. Then $x \in \bigcap_{1\leqslant i<\infty} A^i(I\cap Q) \subseteq \bigcap_{1\leqslant i<\infty} A^i I = B$. Therefore $Q \subseteq \langle A\rangle \cup B \cup \langle A\rangle P$. $\square$

With assumptions (a) and (b) of Proposition 3.3 and assuming $A \neq \emptyset$ we have $B = \bigcap_{1\leqslant i<\infty} A^i B \subseteq \bigcap_{1\leqslant i<\infty} A^i Q \subseteq \bigcap_{1\leqslant i<\infty} A^i S = B$ (3.1, 3.2), hence

$$AB = B = \bigcap_{1\leqslant i<\infty} A^i I = \bigcap_{1\leqslant i<\infty} A^i Q = \bigcap_{1\leqslant i<\infty} A^i S.$$

Combined with $A \cup AS = Q$ these assumptions yield

$$B \cap \langle A\rangle P = \{0\}$$

and

$$B \cup \langle A\rangle P = I \cap Q \text{ (3.3 and 3.4).}$$

$T = \langle A\rangle \cup \{0\}$ is a free semigroup with 0 and we have the decomposition (0-disjoint union)

$$S = T \cup B \cup TP \cup P.$$

Observe that this equation trivially holds if $A = \emptyset$ (with $T = B = TP = \{0\}$).

We also have

$$aS \cap bS = \{0\} \quad \text{for all } a,b \in A \text{ such that } a \neq b,$$

because $\langle A\rangle$ is a free semigroup and $\langle A\rangle \cap I = \emptyset$. The latter condition also shows that T coincides with the Rees factor semigroup S/I.

If S is right projective then every maximal indecomposable right ideal (i.e. maximal indecomposable as a right S-object) that is contained in Q is isomorphic with S^1_S, hence of the form $\{a\}\cup aS$ with $a \notin aS$—this implies $A\cup AS = Q$—and with a left cancellable in S, whereas every maximal indecomposable right ideal contained in P is isomorphic with an idempotent-generated right ideal. The necessity of the following conditions is therefore obvious.

Theorem 3.5. *S is right projective iff*
(a) *A is a set of left cancellable elements,*
(b) *$aI \cap bI = \{0\}$ for all $a,b \in A$ such that $a \neq b$,*
(c) *every maximal indecomposable right ideal is a principal right ideal, and*
(d) *every maximal principal right ideal R with $R = RS$ is isomorphic with an idempotent-generated right ideal.*

Proof. The sufficiency of the conditions depends on whether Q is free over S. Since $a \notin AS$ for every $a \in A$ (by definition of A) and $\{a\} \cup aS$ is free by (a) it remains to be seen that Q is the 0-disjoint union of the $\{a\} \cup aS$ $(a \in A)$. For this observe that $A\cup AS = Q$ by (c) and $aI\cap bI = \{0\}$ as well as $a\langle A\rangle \cap b\langle A\rangle = \emptyset$ and $a\langle A\rangle \cap bI \subseteq \langle A\rangle \cap I = \emptyset$ for $a \neq b, a, b \in A$, by Proposition 3.3. $\square$

We have shown that every right projective semigroup S is a 0-disjoint union of a subsemigroup T, which is a free semigroup with 0, and of the right ideals $B = TB, TP$, and P, and that I is the coproduct of these right ideals. Moreover, as follows from conditions (a) and (b), TP is the coproduct of the right ideals wP ($w \in \langle A\rangle$) and the maximal indecomposable subobjects of TP are isomorphic with idempotent-generated principal right ideals of S. The same holds for the maximal indecomposable subobjects of P, which coincide with the maximal principal right ideals of S contained in P. We have nothing new if $S = S^2$, i.e. $S = P$. To some extent, the structure of this type of right projective semigroup can be clarified by a certain Rees matrix construction [10]. Different special cases of this construction have been considered in [9] and by Fountain and Gould in [7,8]. Especially, every Rees matrix semigroup S over a monoid with 0 is right (and left) projective provided that $S = S^2$ ([9, 3.6]). Steinfeld's [17] and Lallement and Petrich's [15] Rees matrix semigroups are special cases of this. We shall meet 0-disjoint unions of such semigroups in Section 6.

For a general right projective semigroup neither P nor $TP \cup P$ need to be right projective semigroups. We do not know whether $TP \cup P$ right projective implies P right projective but the following example shows that the converse does not hold. We will see below (Proposition 3.6) that in the case $P = P^2$ either none or both of P and $TP \cup P$ are right projective semigroups with 0, but TP may well be not right projective in the latter case (we may modify S by putting $B = \{0\}$ and $P \cdot T = \{0\}$).

Example. Let S be generated by the four-element set $\{a, e, b, c\}$ under the relations $ae = e, e^2 = e, ea = e, eb = b, ec = b, be = b, ce = c$. Then $A = \{a\}, B = \{e\} \cup \langle b\rangle \cup \{0\}, P = \langle c\rangle \cup \{0\}$, and $TP = \langle A\rangle P = \{a^m c^n\colon m, n \geqslant 1\} \cup \{0\}$. S is right projective and P is a free semigroup with 0, hence right projective (Theorem 3.5). We have $\bigcap_{1 \leqslant i < \infty} (TP \cup P)^i = \{0\}$, but $\langle\{a\}\rangle\langle\{c\}\rangle \cup \langle\{c\}\rangle$ is not a free semigroup, hence $TP \cup P$ is not right projective. $\square$

Let R be a right ideal in an arbitrary S and assume that R is isomorphic with eS where e is an idempotent, i.e. $R = uS$, $ue = u$, and $es \mapsto ues$ is one–one, for some u. Let V be a right ideal and $e \in yV$, $e = yv$ with $v \in V$, say. Then $e' = vey$ is an idempotent in V and we have $uy = uye'$, $uyS = R$ and $e's \mapsto uye's$ is one–one, hence R is isomorphic with an idempotent-generated principal right ideal contained in V. For the purposes of this paper we will denote this by $R \preceq V$ (a relation on the right ideals of S, with S fixed). We have seen that $R \preceq yV$ implies $R \preceq V$. If $R \preceq V$ and $V \preceq W$ then $V = vW$ for some $v \in V$, hence $R \preceq W$, i.e. the relation $\preceq$ is transitive.

Now let S be right projective and let R be a maximal principal right ideal contained in P, i.e. $R \in \mathscr{M}(P_S)$. Then $R \preceq I$, and either $R \preceq B$ or $R \preceq P$ or both (because $R \preceq TP$ implies $R \preceq P$). It is well known that eS and fS, e and f idempotents, are isomorphic iff both idempotents belong to the same $\mathscr{D}$-class. As a consequence, an $R \in \mathscr{M}(P_S)$ cannot satisfy both $R \preceq B$ and $R \preceq P$ if S is such that B is saturated by $\mathscr{D}$. Especially, we have either $R \preceq B$ or $R \preceq P$ if at least one of B or $TP \cup P$ is an ideal (equivalently, if $PB = \{0\}$ or $BP = \{0\}$), because $\mathscr{D} \subseteq \mathscr{J}$. In the following we consider the general case where possibly both $R \preceq B$ and $R \preceq P$ for some $R \in \mathscr{M}(P_S)$.

The next proposition is a consequence of 3.5, of the property that $R \preceq yV$ implies $R \preceq V$, and of the fact that the idempotent-generated principal right ideals contained in a right ideal V coincide with the idempotent-generated principal right ideals of the semigroup V. For X a subset of $\mathscr{M}((TP \cup P)_S)$ let V_X denote the union of all $R \in X$. Observe that $\mathscr{M}(P_S) \subseteq \mathscr{M}((TP \cup P)_S)$ and that, for S right projective, $\mathscr{M}((TP \cup P)_S)$ is the disjoint union of $\mathscr{M}((TP)_S)$ and $\mathscr{M}(P_S)$.

Proposition 3.6. *Suppose S is right projective.*

(a) *Let X_1 be a subset of $\mathscr{M}((TP \cup P)_S)$ such that $R \preceq B$ for every $R \in X_1$. Then $T \cup B \cup TV_{X_1} \cup V_{X_1}$ is a right projective semigroup with* 0.

(b) *Let Y_2 be a subset of $\mathscr{M}(P_S)$. Then V_{Y_2} is a right projective semigroup with* 0 *satisfying $V_{Y_2} = (V_{Y_2})^2$ iff $R \preceq V_{Y_2}$ for every $R \in Y_2$.*

(c) *Let Y_2 be a subset of $\mathscr{M}(P_S)$ and, in addition, assume $V_{Y_2} = V_{Y_2}^2$. Let X_2 be a subset of $\mathscr{M}((TP \cup P)_S)$ such that $Y_2 \subseteq X_2$ and such that every $\tilde{R} \in X_2 \backslash Y_2$* satisfies *$\tilde{R} = wR$ for some $w \in T$ and $R \in Y_2$.*

Then V_{X_2} is a right projective semigroup with 0 *iff V_{Y_2} is a right projective semigroup with* 0.

It can be shown that $V_{Y_2} = V_{Y_2}^2$ does not imply the right projectivity of V_{Y_2} over itself.

For $X_1 = \emptyset$ in (a) we have the subsemigroup $T \cup B$ which has the following characterization.

Proposition 3.7. *If S is right projective then $T \cup B$ is the largest subsemigroup S_1 with* 0 *such that $S_1 \backslash S_1^2 \subseteq T$ and S_1 is a right free semigroup with* 0.

Proof. Let U be a subsemigroup which is a right free semigroup with 0 such that $U \backslash U^2 \subseteq T$. Then $U = \langle U \backslash U^2 \rangle \cup \bigcap_{1 \leqslant i < \infty} \langle U \backslash U^2 \rangle^i (\bigcap_{1 \leqslant j < \infty} U^j)$ (by 3.4 applied to U and because $P(U_U) = \{0\}$), hence $U \subseteq T \cup B$. The subsemigroup $T \cup B$ is right projective by 3.5 (because $(T \cup B) \backslash (T \cup B)^2 = A$ and $A \cup A(T \cup B) \cup \{0\} = A \cup AT \cup B = T \cup B$) and satisfies $P((T \cup B)_{T \cup B}) = \{0\}$, hence $(T \cup B)$ is right free. $\square$

In an arbitrary S, put $Y' = \{R \in \mathscr{M}(P_S)\colon R \preceq B\}$ and $S' = T \cup B \cup TV_{Y'} \cup V_{Y'}$, and, furthermore, $Y'' = \mathscr{M}(P_S) \backslash Y'$ and $S'' = TV_{Y''} \cup V_{Y''}$. S' is a subsemigroup and S'' a right ideal of S. $T \cup S''$ and $T \cup B \cup S''$ are subsemigroups of S.

If we assume that S is right projective then the set Y'' satisfies $R \preceq V_{Y''}$ for all $R \in Y''$, hence for all $R \in \mathscr{M}(S''_S)$ (because $\{wR\colon R \in \mathscr{M}(V_{(Y'')S}), w \in \langle A \rangle\} = \mathscr{M}((TV_{Y''})_S)$), and wR is isomorphic with R for every right ideal R). Thus $S''^2 = S''$ and S is the 0-disjoint union of the right projective semigroups S' and S'' (3.6).

We have chosen Y' as large as possible with respect to the property that $R \preceq B$ for all $R \in Y'$ but there may be other partitions of $\mathscr{M}(P_S)$ (and still other partitions of $\mathscr{M}((TP \cup P)_S))$ that lead to right projective semigroups with 0. Especially, if $Y_2 \subseteq \mathscr{M}(P_S)$ is such that $R \preceq V_{Y_2}$ for all $R \in Y_2$ and $Y'' \subseteq Y_2$ then, putting $Y_1 = \mathscr{M}(P_S) \backslash Y_2$, S is the 0-disjoint union of the semigroups $T \cup B \cup TV_{Y_1} \cup V_{Y_1}$ and $TV_{Y_2} \cup V_{Y_2}$, both right projective. Observe that there is a largest set $\bar{Y}'' \subseteq \mathscr{M}(P_S)\}$ such that $R \preceq V_{\bar{Y}''}$ for all $R \in \bar{Y}''$, namely the union of all subsets with that property.

We can characterize the right projectivity of S in terms of the subsemigroups S', S'' and $T \cup S''$.

Proposition 3.8. *S is right projective iff $S = S' \cup S''$, $S' \cap S'' = \{0\}$, $S''^2 = S''$, and both S' and $T \cup S''$ are right projective.*

Proof. Let S be right projective. Then S' is right projective and it remains to be seen that $T \cup S''$ is right projective. Since $(T \cup S'') \backslash (T \cup S'')^2 = T \backslash T^2 = A$ and $A \cup A(T \cup S'') \cup \{0\} = T \cup TV_{Y''}$ and since $V_{Y''}$ is right projective and satisfies $V_{Y''} = V_{Y''}^2$ (3.6) this follows from Theorem 3.5.

Conversely, consider an $R \in \mathscr{M}(P_S)$. If $R \subseteq S'$ then R is isomorphic with an idempotent-generated principal right ideal (of S, and of S', by definition of S'). If $R \subseteq S''$ then R is a cofactor of $(T \cup S'')_{(T \cup S'')}$. As $S''^2 = S''$ we have $RS'' = R$, hence R is a cofactor of $\tilde{P} = P((T \cup S'')_{(T \cup S'')})$. As $T \cup S''$ is right projective R is a 0-disjoint union of right ideals of $T \cup S''$ that are isomorphic with idempotent-generated principal right ideals of $T \cup S''$, hence of S. It follows that R is isomorphic with a single idempotent-generated principal right ideal of S which is contained in S'' (because T, a subsemigroup of $T \cup S''$ that is free with 0, does not contain a non-zero idempotent). Now $S = S' \cup S''$ implies $S = T \cup B \cup TP \cup P$. Therefore every maximal indecomposable right ideal not contained in P is contained in $T \cup TS = A \cup AS \cup \{0\}$. Hence (c) and (d) of Theorem 3.5 are fulfilled.

We have $A \subseteq S' \backslash S'^2$ and $A = (T \cup S'') \backslash (T \cup S'')^2$. By the right projectivity of both S' and $T \cup S''$ we have that every $a \in A$ is left cancellable in S' and left cancellable in $T \cup S''$ and that $a \neq b, a, b \in A$ implies $aS' \cap bS' = \{0\}$ and $a(T \cup S'') \cap b(T \cup S'') = \{0\}$. It remains to be seen (Theorem 3.5) that $a \in A$ is left cancellable in S and that $aI \cap bI = \{0\}$ if $a \neq b$. Assume $ax = ay$ with $a \in A, x \neq y, y \in S''$ (so that $x \in S'$ and $ay \neq 0$) or $bx = ay \neq 0$ with $a, b \in A, a \neq b, y \in S''$ (so that $x \in S'$). Then ay is a non-zero element of $S' \cap S''$ contrary to the 0-disjointness of S' and S''. □

Finally a characterization of the right projectivity of S in terms of the subsemigroups S', S'', and $T \cup B \cup S''$.

Proposition 3.9. *S is right projective iff $S = S' \cup S''$, $S' \cap S'' = \{0\}$, and both S' and $T \cup B \cup S''$ are right projective.*

Proof. If S is right projective then S is the 0-disjoint union of $T \cup B$ and $TP \cup P$ (cf. the remarks following Theorem 3.5) and S' and S'' are right projective. The subsemigroup $U=T\cup B\cup S''$ is right projective because $T\cup B$ is right projective (3.7), $T=\langle U\backslash U^2\rangle\cup\{0\}$, $B=\bigcap_{1\leqslant j<\infty}T^j(\bigcap_{1\leqslant i<\infty}U^i)$ (because $TB=B$ by 3.3), and $U''=S''$ so that Proposition 3.8 can be applied (to S for the right projectivity of $T\cup S''$ and to U instead of S in the other direction).

Conversely, consider $S'=T\cup B\cup TV_{Y'}\cup V_{Y'}$ and $S''=TV_{Y''}\cup V_{Y''}$ and put $U=T\cup B\cup S''$. We will use $S=S'\cup U$ and the right projectivity of U to show that every $R\in Y''$ belongs to $\mathscr{M}(P(U_U))$. Clearly, R is a cofactor of U_U and $R=RS$ implies $R=RU$ (if $r=ys$, with $y\in R$, then $s\in U$ or $s\in zeS$ for some $z\in TV_{Y'}\cup V_{Y'}$ and $e\in B$, hence $r=yz\cdot ex, yz\in R, ex\in U$). It follows that R is a cofactor of $P(U_U)$, hence a 0-disjoint union of principal right ideals of U that are isomorphic with idempotent-generated principal right ideals of U. Now $A\subseteq U\backslash U^2$ implies that any idempotent-generated principal right ideal of U is contained in the right ideal $B\cup S''$ of S so that it is an idempotent-generated principal right ideal of S. R being indecomposable we have that R is isomorphic with a single idempotent-generated principal right ideal of U.

Observe that R is not isomorphic with an idempotent-generated right ideal contained in B. It follows that every $R\in Y''$ is isomorphic with an idempotent-generated principal right ideal contained in $TV_{Y''}\cup V_{Y''}$, hence with an idempotent-generated principal right ideal contained in $V_{Y''}$. This implies $V_{Y''}^2=V_{Y''}$, hence $S''^2=S''$. We now apply 3.5 to the semigroup $T\cup S''=T\cup TV_{Y''}\cup V_{Y''}$ (using the aforementioned properties of $V_{Y''}$ and again the right projectivity of U). Hence $T\cup S''$ is right projective and we can apply the only-if part of 3.8. □

It can be shown that none of the above conditions can be omitted. Especially, it does not suffice to require that $T\cup S''$ is right projective (instead of $T\cup B\cup S''$).

For a right projective semigroup S, we have $A=S'\backslash S'^2=U\backslash U^2$, where $U=T\cup B\cup S''$. Moreover, $\bigcap_{1\leqslant i<\infty}S'^i=S'\cap I$, $\bigcap_{1\leqslant i<\infty}U^i=U\cap I$, $\bigcap_{1\leqslant j<\infty}T^j(S'\cap I)=\bigcap_{1\leqslant j<\infty}T^j(U\cap I)=B$, and (with an extension of notation) $(S')'=S'$, $(S')''=\{0\}$, $U''=(S'')''=S''$, and $U'=T\cup B$. Thus a repetition of the decomposition procedure leads to nothing new.

It is easy to show that any right projective semigroup S is completely determined by its subsemigroups S', $T\cup S''$, and I, or alternatively, by S', $U=T\cup B\cup S''$, and P. If $PB=P$ (equivalently: $(P\cap S'')B=P\cap S''$) then S is determined by S' and U alone, but not in general.

Example. Let S be generated by the five-element set $\{a,e,u,f,g\}$ under the relations $ae=ea=e^2=e$, $ue=u$, $fa=f^2=f$, $fe=fg=ga=g$, $eu=u^2=0$, $ef=0$, and $fu=g$. Then $A=\{a\}$, $B=\{e,0\}$, $TV_{Y'}\cup V_{Y'}=\{a^mu\colon m\geqslant 1\}\cup\{u,0\}$, and $TV_{Y''}\cup V_{Y''}=\{a^mf\colon m\geqslant 1\}\cup\{a^mg\colon m\geqslant 1\}\cup\{f,g,0\}$.

Let $\tilde{S}$ be generated by the same five-element set under the same relations except that $fu=g$ is replaced by $fu=0$. Then both S and $\tilde{S}$ are non-isomorphic right projective semigroups but the identity mapping induces isomorphisms from the subsemigroups S' and $T\cup B\cup S''$ of S to the corresponding subsemigroups of $\tilde{S}$.

In the final part of this section we discuss to what extent the components T, B, $S' \cap P$, and $S'' \cap P$ of a right projective semigroups S can be chosen independently. Especially, we discuss the alternatives for composing S' and $T \cup S''$, T and $S'' \cap P$, $T \cup B$ and $B \cup (S' \cap P)$, and B and $S' \cap P$.

Given right projective semigroups S_1 and S_2 that satisfy $S_1 \cap S_2 = \langle S_i \backslash S_i^2 \rangle \cup \{0\}$, $i = 1, 2$, $S_1 = S_1'$, and $\bigcap_{1 \leqslant i < \infty} S_2^i = S_2''$, there are generally various possibilities to build a right projective semigroup S such that $S' = S_1'$ and $S'' = S_2''$ (so that $T = S_1 \cap S_2$ and $B = B_1$) as the foregoing example shows.

We can always define $B_1 S_2'' = \{0\}$, which forces $I_1 S_2'' = \{0\}$, and $S_2'' B_1 = \{0\}$, which forces $S_2'' I_1 = \{0\}$. If $S_2'' T^k = \{0\}$ for some $k \geqslant 1$ then $B_1 S_2'' \subseteq \{x \in B_1 \colon xB_1 = \{0\}\}$ (because $B_1 = TB_1$). Therefore, if $S_2'' T^k = \{0\}$ and B_1 has no zero divisors then the mentioned composition is the only one possible.

There is a straightforward way to construct S'' from $V = S'' \cap P$ $(= V_{Y''})$ and T. Choose an arbitrary homomorphism $h : w \mapsto h_w$ from $\langle A \rangle$ into $End_V V$ (the semigroup of right translations of V) and define

$$TV = T \times V \backslash ((\{0\} \times V) \cup (\langle A \rangle \times \{0\})) \cup \{0\},$$

$$w \cdot v = (w, v) \text{ and } v \cdot w = vh_w \ (w \in \langle A \rangle,\ v \in V \backslash \{0\}),$$

the other products are uniquely determined thereby. Different homomorphisms lead to different semigroups and there are no other possibilities (up to isomorphism). Correspondingly S' is obtained from $T \cup B$ and $B \cup (S' \cap P)$ (here the homomorphism h from T to $End_{S' \cap P}(S' \cap P)$ is uniquely determined by the right action of T on B).

As will be discussed in the next section, T and B cannot be chosen independently. As the foregoing construction of S'' from $S'' \cap P$ shows, T and S'' are also not independent, and the same holds for T and $S' \cap I$. At this point we consider how $S' \cap P$ is related to B.

Let S be right projective and put $V_1 = S' \cap P$ $(= V_{Y'})$. Then the semigroup $B \cup V_1$ is an extension of the semigroup B by a projective right B-object V_1 such that $V_1 = V_1 B$ (0-disjoint from B). Thus the semigroup $B \cup V_1$ depends on the right multiplication of V_1 on B. Any Rextension of B is appropriate, i.e. if $B \cup \tilde{V}_1$ is another extension by a projective right B-object $\tilde{V}_1$ such that $\tilde{V}_1 = \tilde{V}_1 B$ then there is a unique way to replace the subsemigroup $TV_1 \cup V_1$ of S by a semigroup $T\tilde{V}_1 \cup \tilde{V}_1$ such that $B \cup V_1$ is replaced by $B \cup \tilde{V}_1$ and such that a right projective semigroup $\tilde{S}$ is obtained; moreover, $\tilde{S}$ is isomorphic to S over the subsemigroup $T \cup B \cup S''$ iff $B \cup \tilde{V}_1$ is isomorphic to $B \cup V_1$ over B.

We will see that the different extensions of B can be described (up to isomorphism over B) by certain equivalence classes of a subset of $End_B B \times E(B)$. Let R be a maximal indecomposable right ideal of $B \cup V_1$ contained in V_1 (hence $R \in Y'$) and let R be isomorphic with eB where $e = e^2 \in B$. The right multiplications $b \mapsto br$ $(b \in B)$ by elements $r \in R$ belong to $End_B B$ which we regard as a right B-object $(\varphi x : b \mapsto b\varphi \cdot x)$. If $q \in R$ is a generator of R (i.e. $R = qB$) then there exists an idempotent $f \in \mathscr{D}e$ (the $\mathscr{D}$-class of e) such that $q = qf$. In this case $x \mapsto qx$ is a bijection from $\mathscr{R}f$ (the $\mathscr{R}$-class of f) to $\mathscr{R}q$ (the $\mathscr{R}$-class of q). Clearly, the right multiplication $\psi : b \mapsto bq$

is such that $\psi f=\psi$. If q' is another generator then there exists an $f'=f'^2\in\mathscr{D}e$ such that $q'=q'f'$. Hence the right multiplication $\psi':b\mapsto bq'$ satisfies $\psi'f'=\psi'$ and there exists an $x\in\mathscr{R}f\cap\mathscr{L}f'$ such that $\psi x=\psi'$. Conversely, if there exists an $x\in\mathscr{R}f\cap\mathscr{L}f'$ such that $\psi x=\psi'$ for some $\psi'\in End_B B$ then $\psi':b\mapsto bq'$ for some generator q' of R.

Now consider the set $\Phi_B=\{(\phi,f)\colon \phi\in End_B B, f=f^2\in B\phi f=\phi\}$ which we call the set of codominated endomorphisms of ${}_BB$. Observe that $B\phi\subseteq Bf$. The set $\{((\phi,f),(\phi',f'))\colon \phi x=\phi'$ for some $x\in\mathscr{R}f\cap\mathscr{L}f'\}$ is an equivalence relation on Φ_B, any class of which will be called a $\mathscr{D}$-related class (of codominated endomorphisms). We have seen before that the right multiplications by generators of R ($\in Y'$) form a $\mathscr{D}$-related class which we denote by $\mathbf{C}_R$.

If C is any $\mathscr{D}$-related class we may choose a pair $(\phi,f)\in\Phi$ and define the right B-object $\mathbf{R}_C=\{(1,y)\colon y\in fB\backslash\{0\}\}\cup\{0\}$ by $(1,y)\cdot b=(1,yb)$ if $yb\neq 0$ and $(1,y)\cdot b=0$ otherwise. Then $\mathbf{R}_C$ is isomorphic with fB and has $\tilde{u}=(1,f)$ as a generator. Moreover (assuming that $\mathbf{R}_C$ is 0-disjoint from B), we may replace R by $\mathbf{R}_C$ in $B\cup V_1$ by defining $b\cdot\tilde{u}y=b\phi\cdot y$ so that we obtain another extension of B. We have that $\mathbf{R}_{(\mathbf{C}_R)}$ is isomorphic with R and $\mathbf{C}_{(\mathbf{R}_C)}=C$.

It follows that every extension of B by a projective right B-object V_1 (hence every subsemigroup $B\cup(S'\cap P)$ of a right projective semigroup S) is determined up to isomorphism over B by a mapping $\theta:Z\to\mathscr{P}(\Phi_B)$ into the powerset of Φ_B (Z an arbitrary set which indexes $Y'=\mathscr{M}(V_1)$) such that $\theta(z)$ is a $\mathscr{D}$-related class of codominated endomorphisms. Mappings $\theta:Z\to\mathscr{P}(\Phi_B)$ and $\tilde{\theta}:\tilde{Z}\to\mathscr{P}(\Phi_B)$ that are not connected by a bijection between $\tilde{Z}$ and Z yield extensions that are not isomorphic over B.

4. Free and right free semigroups with 0

In this section let S be a semigroup with 0 which is a 0-disjoint union of a subsemigroup $\tilde{T}$ such that $\bigcap_{1\leqslant i<\infty}\tilde{T}^i=\{0\}$ and a two-sided ideal $\tilde{B}$ such that $\tilde{B}\subseteq S^2$. The condition on $\tilde{T}$ implies that $\tilde{T}$ is generated by $\tilde{T}\backslash\tilde{T}^2$ (as a semigroup with 0) and the condition on $\tilde{B}$ implies that $\tilde{T}\backslash\tilde{T}^2=S\backslash S^2$ and that $\tilde{B}=\tilde{T}\tilde{B}\cup\tilde{B}\tilde{B}\cup\tilde{B}\tilde{T}$ which implies $\tilde{B}=\bigcap_{1\leqslant i<\infty}S^i$. Hence, retaining our previous notation, we have $\tilde{T}=T$ and $\tilde{B}=I$.

Thus we consider the case that T and I are 0-disjoint and that $\bigcap_{1\leqslant i<\infty}T^i=\{0\}$. These conditions hold in any right projective semigroup S (3.5, 3.3). As before, we denote the set $S\backslash S^2=T\backslash T^2$ by A and the right ideal $\bigcap_{1\leqslant i<\infty}T^iI$ by B.

If $I\neq\{0\}$ then S is right free iff

(a) $TI=I$,

(b) $aI\cap bI=\{0\}$ for all $a,b\in T\backslash T^2$ with $a\neq b$, and

(c) $c\mapsto vc$ ($c\in I$) is injective for every non-zero $v\in T$.

For if S is right free, hence right projective, then $I=B$ (3.7) and the conditions follow from 3.5 and 3.3. Conversely, observe that conditions (b) and (c) together guarantee that T is a free semigroup with 0 (3.1), because the assumption on T forbids that T is a finite cyclic semigroup with an adjoined 0. Furthermore, the freeness of T

together with (a) and (b) yields that every maximal indecomposable right ideal R is principal and satisfies $RS \subset R$, and that all $a \in T \setminus T^2$ are left cancellable. (a) implies $I = B$. Hence S is right projective (3.5) and therefore right free (3.7).

If T is free we may assume that $T \setminus \{0\}$ is the standard free semigroup A^+ that consists of the non-empty finite sequences over the set A (where A also denotes the set of sequences of length 1). The monoid of all finite sequences over A will be denoted by A^*. We will also use the infinite sequences over A, i.e. the set A^ω which is a left A^*-set. The relation $v = \{(\sigma, \tau)\colon \sigma, \tau \in A^*\zeta \text{ for some } \zeta \in A^\omega\}$ is a congruence of the left A^*-set A^ω; the congruence class of σ with respect to v will be denoted by $\bar{\sigma}$ ($\bar{\sigma}$ is a maximal indecomposable left A^*-set). The A^*-set A^ω/v will be denoted by $\overline{A^\omega}$. Occasionally, we will denote an arbitrary element of $\overline{A^\omega}$ by Σ.

Every semigroup I with 0 is appropriate for an extension to a right free semigroup: Choose a pair (λ, ρ) from the translational hull of I such that λ is an automorphism of I_I (e.g. $\lambda = \rho = \Delta_I$) and define $ab = \lambda b$, $ba = b\rho$ where a is the generator of an infinite cyclic semigroup, hence $T = \langle\{a\}\rangle \cup \{0\}$. The semigroups thus obtained are characterized by the following statement.

Proposition 4.1. *If $S \neq \{0\}$ then the following are equivalent*:

(1) *S_S is free and indecomposable.*

(2) *S_S is projective and indecomposable, and S is an ideal extension of a semigroup I by an infinite cyclic semigroup with* 0.

(3) *S_S is projective and indecomposable, and $S^2 \neq S$.*

Proof. (1) ⇒ (2) follows from (a) and (b) and the freeness of T, (2) ⇒ (3) is clear, and (3) ⇒ (1) follows from the 0-disjointness of Q and P (as defined at the beginning of the preceding section). □

Corollary 4.2. *Let $\tilde{S}$ be a right projective semigroup with* 0 *and suppose that $\tilde{S}_{\tilde{S}}$ is indecomposable. Then $\tilde{S} = \tilde{S}^2$ or $\tilde{S}$ is an ideal extension of a semigroup with* 0 *by an infinite cyclic semigroup with* 0.

The class of right indecomposable right projective non-right free semigroups $\tilde{S}$ (i.e. case $\tilde{S} = \tilde{S}^2$ in the statement) properly contains the class of semigroups with 0 having a left identity element. A structural characterization of the larger class is not yet known.

Assume that S is right free. For any $\sigma \in A^\omega$ let B_σ be the intersection of the right ideals vS where $v \in A^+$ and $\sigma \in vA^\omega$. We have $B_\sigma = \bigcap_{\sigma \in vA^\omega} vB$ (because A^+ is free and $A^+ \cap B = \{0\}$). B_σ may be the zero ideal but it follows from (a), (b), and (c) that every non-zero $c \in B$ is contained in B_σ for a unique $\sigma \in A^\omega$. For $\sigma \neq \tau$ we have $B_\sigma \neq B_\tau$ or $B_\sigma = B_\tau = \{0\}$ (by (b) and (c)), hence B is the 0-disjoint union of the right ideals B_σ ($\sigma \in A^\omega$).

Consider an $R \in \mathscr{M}(B_S)$. Then $R \in \mathscr{M}(B_\sigma)$ for a unique $\sigma \in A^\omega$, $wR \in \mathscr{M}(B_{w\sigma})$ for every $w \in A^+$ (because $r \mapsto wr$ is a bijection from B_σ to $B_{w\sigma}$), and if $\sigma \in vA^\omega$, $\sigma = v\tau$ say, then $v^{-1}R = \{c \in I\colon vc \in R\} \in \mathscr{M}(B_\tau)$ (by the same reason as before). As a consequence, if $\sharp A > 1$ there are at least $\sharp A^+$ maximal indecomposable subobjects of B_S

that are isomorphic with R, i.e. at least $\aleph_0$ such subobjects if A is finite or countably infinite and at least $\sharp A$ such subobjects if A is uncountably infinite.

Remark 4.3. Let S be right free and suppose that there is a non-zero ideal U contained in B which is a 0-disjoint union of finitely many indecomposable right ideals of S. Then $\sharp A = 1$, i.e. S is an ideal extension of a semigroup by an infinite cyclic semigroup with $\{0\}$.

For any $R \in \mathscr{M}(B_\sigma)$ let $\tilde{R}$ denote the right ideal $\bigcup_{v,w\in A^+,\sigma\in vA^\omega} wv^{-1}R$. Then $T\tilde{R} \subseteq \tilde{R}$ (trivially), and $\tilde{R}$ is maximal among the right ideals that are indecomposable into right ideals that are left T-objects). B is the 0-disjoint union of the right ideals $\tilde{R}$ with $R \in \mathscr{M}(B_S)$. Possible coarser cofactorizations of B into right ideals that are left T-objects are given by the right ideals of the form $\bar{R}$, union of all $U \in \mathscr{M}(B_S)$ isomorphic with R, and by the right ideals of the form $B_{\bar{\sigma}}$, join of all $\tilde{U}$ with $U \in \mathscr{M}(B_\sigma)$ (observe that these right ideals depend only on the congruence class $\bar{\sigma}$ of σ). For every $R \in \mathscr{M}(B_S)$, $S_R = T \cup \tilde{R}$ is a subsemigroup which is right free and satisfies $\bigcap_{1\leqslant i<\infty} S_R^i = \tilde{R}$.

Consider the coproduct $B = \coprod_{\Sigma\in\overline{A^\omega}} B_\Sigma$. A sequence $\sigma \in A^\omega$ is called rational (or ultimately periodic) if there exist $v, w \in A^+$ and $\tau \in A^\omega$ such that $v\tau = \tau$ and $\sigma = w\tau$ (in short: $\sigma = wv^\omega$), and irrational otherwise. Clearly, every rational sequence σ has a set of shortest periods v, all of which are conjugate [14], and every $\tau \in \bar{\sigma}$ is rational with the same set of shortest periods. Observe that B_Σ ($\Sigma \in \overline{A^\omega}$) is such that $vR = R$ for some $v \in A^+$ and some $R \in \mathscr{M}(B_\Sigma)$ iff Σ is a rational class (clearly, $vR = R$ iff $R \in B_{v^\omega}$).

The following statement is a consequence of conditions (a)–(c) above.

Remark 4.4. Let I be a 0-disjoint union of right ideals I_i of S, $i \in \Gamma$, such that $TI_i \subseteq I_i$ for every $i \in \Gamma$. Then S is right free iff every subsemigroup $T \cup I_i$ is right free and satisfies $TI_i = I_i$.

The following is again an immediate consequence of (a)–(c).

Remark 4.5. Let U be an ideal of S contained in I.
(1) If S is right free then S/U is right free iff $a^{-1}U = U$ for every $a \in T\backslash T^2$.
(2) If $a^{-1}U = U$ for every $a \in T\backslash T^2$ then S is right free iff S/U and $T \cup U$ are right free.

Observe that the right socle, i.e. the join of the 0-minimal right ideals, of a right free semigroup S, and every component thereof (i.e. principal ideal of S contained in the right socle), is an ideal U as considered in 4.5.

If S is right free then the maximal indecomposable right ideals of the semigroup $I = B$ are not too different from the maximal indecomposable subobjects of B_S. Indeed, if $R \in \mathscr{M}(B_B)$ and $v \in T$ then $Rv\backslash R$ is contained in $\{0\}B^{-1} = \{s\colon sB = \{0\}\}$. This is then an ideal U of S that satisfies the condition of Remark 4.5. We now consider the case where $U = \{0\}B^{-1} = B$, i.e. $B^2 = \{0\}$.

Let $T = A^+ \cup \{0\}$ and let $q : \Sigma \mapsto q_\Sigma$ be a mapping from $\overline{A^\omega}$ into the class of (left-right) $T - T$-biobjects (cf. Section 6). Assume that $m \mapsto am$ $(m \in q_\Sigma)$ is an automorphism of $(q_\Sigma)_T$, for every $a \in A$ $(\Sigma \in \overline{A^\omega})$. For any $\sigma \in \Sigma$ define $q_\sigma = (\{\sigma\} \times q_{\bar\sigma}) \backslash \{(\sigma, 0)\} \cup \{0\}$, with $(\sigma, m) \cdot t = (\sigma, mt)$ if $mt \neq 0$ and $(\sigma, m) \cdot t = 0$ otherwise, so that $(q_\sigma)_T$ is isomorphic with $(q_{\bar\sigma})_T$.

Let S_q be the coproduct of T_T and of the right T-objects q_σ (we assume that T and $\coprod_{\sigma \in A^\omega} q_\sigma$ are 0-disjoint) and extend the multiplication to all of $S_q \times S_q$ by

$$v(\sigma, m) = (v\sigma, vm) \quad (v \in A^+, m \in q_{\bar\sigma})$$

and

$$(\sigma, m_1)(\tau, m_2) = 0 \quad (m_1 \in q_{\bar\sigma}, m_2 \in q_{\bar\tau}).$$

Then S_q is right free and $T = S/I$ $(I = \bigcap_{1 \leqslant i < \infty} S^i = B)$. We will instantly see that every right free S with $B^2 = \{0\}$ is obtained in this way (up to isomorphism).

For this purpose we restrict the admissible mappings q a bit. Fix a well-order on A and let $q : \Sigma \mapsto q_\Sigma$ be such that $m \mapsto am$ $(m \in q_\Sigma)$ is the identity automorphism of $(q_\Sigma)_T$ except possibly when a is the first element in the well-order that appears in a period of Σ (i.e. a is the first element such that $\sigma = (av)^\omega$ for some $\sigma \in \Sigma$ and some $v \in A^*$). Every right free S with $B^2 = \{0\}$ is of the form S_q: choose $q_\sigma = B_\sigma$ for some $\sigma \in \Sigma$—arbitrarily if Σ is irrational, but otherwise such that $\sigma = (av)^\omega$ with a as before and, taking v as short as possible, such that $(a, m) \mapsto avm$ is the scalar multiplication of the left T-object q_Σ. Moreover, for such mappings p and q, S_p and S_q are isomorphic over T if and only if for every $\Sigma \in \overline{A^\omega}$ there exists a $T - T$-isomorphism of p_Σ to q_Σ.

We would like to know whether a given I allows right free extensions by non-cyclic free semigroups with 0, but have not found conditions that go beyond an interpretation of conditions (a)–(c) in terms of the translational hull of I. In cases where the translational hull can explicitly be determined there is a good chance to find an answer. E.g. if I is the semigroup of $K \times K$-matrices with at most one non-zero entry over a group with 0 (the Brandt semigroup) where K is infinite, then there exist right free extensions of I by a free T if and only if $\sharp(T \backslash T^2) \leqslant \sharp K$. If I is the Rees matrix semigroup $(G^{\underline{0}}, p)$ where G is a group and $p : \Lambda \times K \to G^{\underline{0}}$ is constant with $p_{\lambda,k} = 1$ and K infinite we have the same result (we write p instead of the usual P in order to have the mapping notation look more natural). Another class of right free semigroups with non-cyclic A^+ will be considered in the next section. We mention one more case where only a single extension is possible.

Proposition 4.6. *If S and I are right free semigroups with* 0 *then $\sharp A = 1$ and $ax = x$ for $a \in A$ and all $x \in I$.*

Proof. Put $\tilde I = \bigcap_{1 \leqslant i < \infty} I^i$. We may assume that I is the disjoint union of C^+, with $C \neq \emptyset$, and $\tilde I$. The maximal indecomposable right ideals of I have the form $\{c\} \cup cI$ with $c \in C$. I is the union of the right ideals $\{ac\} \cup acI$ of I $(a \in A, c \in C)$ all of which are pairwise 0-disjoint (conditions (a)–(c)). Therefore, if $a \in A$, then ac belongs to C for every $c \in C$. It follows that ca belongs to C^+. Hence the element $(\lambda : b \mapsto ab, \rho : b \mapsto ba)$ of the translational hull of I induces an element of the translational hull of C^+. It is easily seen that the translational hull of C^+ is C^*. Therefore $ac = c$ (and $ca = c$) for every $c \in C$, hence $ab = b$ for every $b \in I$ because of $\tilde{I} = C\tilde{I}$. Now property (b) implies $\sharp A = 1$. $\square$

At the end of this section we consider cancellativity, finiteness, and homogeneity conditions that cause a right free semigroup with 0 to be free as a semigroup with 0. A semigroup with 0 will be called right 0-cancellative if $xs = ys \neq 0$ implies $x = y$. A semigroup with 0 is 0-cancellative if it is left and right 0-cancellative. Clearly, if H is a cancellative semigroup then $H^{\underline{0}}$ is 0-cancellative. A partially ordered set with a least element is said to satisfy the restricted minimal condition if any non-empty subset that is non-trivially bounded from below contains a minimal element. We apply this property to the principal right and to the principal two-sided ideals of a semigroup with 0.

Theorem 4.7. *The following are equivalent*:
(0) *S is a free semigroup with* 0.
(1) $S^2 \neq S$, *S is* 0-*cancellative*, *S_S projective*, *and I_S finitely generated.*
(2) *S is* 0-*cancellative*, *S_S free*, *and I_S finitely generated.*
(3) *S is right* 0-*cancellative*, *S_S projective*, *B_S finitely generated*, *and $K^2 \neq K$ for every non-zero ideal K.*
(4) *S is right free and satisfies the restricted minimal condition for principal right ideals.*
(5) *S is right free and satisfies the restricted minimal condition for principal ideals.*
(6) *S is right* 0-*cancellative and S_S and B_B are free.*
(7) *S_S and ${}_SI$ are free.*

Proof. It is easily checked that a free semigroup with 0 satisfies (1)–(7).

(1) ⇒ (2): $S^2 \neq S$ implies $A \neq \emptyset$ and the finite generatedness of I_S $(= B \cup A^+P \cup P)$ implies $P = \{0\}$.

We now show that (2) as well as (3) implies the following.

(3′) S is right 0-cancellative, S_S is free, B_S finitely generated, and either $B = \{0\}$ or $B^2 \neq B$.

(2) ⇒ (3′): Any non-zero generator b of a maximal principal right ideal within B (which exists by finite generatedness provided that $B \neq \{0\}$) does not belong to B^2, and not even to BS because $b = bs$ would imply $s = s^2$, hence $as = as^2$, i.e. $a = as \in B$ for any $a \in A$.

(3) ⇒ (3′): If $P \neq \{0\}$ then $K = BP \cup TP \cup P$ is a non-zero ideal such that $K^2 = K$ which is excluded by assumption. Therefore $P = \{0\}$ so that S_S is free and B is an ideal, hence $B^2 \neq B$ or $B = \{0\}$.

(3′) ⇒ (0): Assume $B \neq \{0\}$. Then $\sharp A = 1$ because $S^2 \neq S$ (so that $A \neq \emptyset$) and B_S is finitely generated (so that $\sharp A \leqslant 1$). Say $A = \{a\}$. Let $\tilde{X}$ be a minimal set of generators of B_S and consider the set $X = \tilde{X} \cap (B \backslash B^2)$. The mapping $y \mapsto ay$ is an isomorphism of B_S, hence induces a permutation on $B \backslash B^2$. It follows that this mapping also permutes the finite set of $\mathscr{R}$-classes $\mathscr{R}x$ $(x \in X)$. Hence for any $x \in X$ there exists an n such that $a^n x \in \mathscr{R}x$, hence $a^n x = xa^k$ and $xa^k \cdot a^l = x$ for some k and l. Thus a^n permutes the finite set $x\{a\}^+$ (which has at most $k + l$ elements) which implies that there exists an m (a multiple of n) such that $a^m x = x$. This implies $a^{2m} = a^m$ by right 0-cancellativity, which is impossible because $\{a\}^+$ is free.

(4), (5) $\Rightarrow$ (0): Assume that there exists a non-zero $b \in B$. Then $b \in B_\sigma$ for some $\sigma \in A^\omega$, i.e. there exists a sequence $v_1, v_2, v_3, \ldots \in A^+$ such that $v_{i+1} \in v_i A^+$ and $b \in v_i S$ for $i = 1, 2, 3, \ldots$. Hence $v_1 S^{\underline{1}}, v_2 S^{\underline{1}}, , v_3 S^{\underline{1}}, \ldots$ is a strictly descending chain of principal right ideals all of which contain $bS^{\underline{1}}$—a contradiction. Correspondingly for principal ideals $S^{\underline{1}} v_1 S^{\underline{1}}, S^{\underline{1}} v_2 S^{\underline{1}}, \ldots$ and $S^{\underline{1}} b S^{\underline{1}}$.

(6) $\Rightarrow$ (0): Since $ax = x$ for all x in B (4.6), so that $a^2 x = ax$, we have $B = \{0\}$ by right 0-cancellativity.

(7) $\Rightarrow$ (0): If $I = B \neq \{0\}$ then $SI \neq I$ because ${}_S B$ is free which contradicts the freeness of S_S which requires $SB = B$. $\square$

From the equivalence of (0) and (7) we deduce the following.

Corollary 4.8. *S is a free semigroup with* 0 *iff every left and every right ideal is free as a left or right S-object, respectively.*

The study of one-sided hereditarily free semigroups (meaning, say, that every right ideal is free as a right S-object, without any conditions on the left side) is postponed to another paper. As a hint to the nature of such semigroups we mention without proof the following special case. As is well known (cf. [16]), an ordinal α (i.e. the set of all ordinals smaller than α) is closed under ordinal addition if and only if α is a power of ω. We mean the operation of ordinal addition when we speak of the semigroup α. Note that Proposition 4.9 requires modification if "semigroup" (e.g. "S is a semigroup with 0", as maintained throughout) is replaced by "monoid" (cf. Section 7.1).

Proposition 4.9. *The following are equivalent*:
(1) $S^{\underline{1}} = \omega^\delta \cup \{0\}$ *for some ordinal* δ.
(2) *All non-zero right ideals of S are isomorphic with* $S^{\underline{1}}_S$.
(3) *Every right ideal of S is free as a right S-object and the right ideals form a chain.*

Corollary 4.10. *Suppose* $S^2 \neq S$. *Then the following are equivalent*:
(1) *S is an infinite cyclic semigroup with* 0.
(2) *All non-zero right ideals of S are isomorphic and the same holds for the non-zero left ideals.*
(3) *Every right ideal of S is free as a right S-object and every left or right ideal is two-sided.*

5. Semigroups of sequences

The well-known representation of a free semigroup as a semigroup of finite-length sequences has already been used in the last section. Now we consider a generalization of such semigroups. The essential point is that concatenation can be extended to arbitrary well-ordered sequences (and in fact to somewhat more general structures). The inclusion of a zero element yields a great diversity of semigroups with 0. All of them turn out to be right free.

We use a standard set theory in which any ordinal α coincides with the set of all smaller ordinals (cf. [16]). In this section "0" will always denote an element which is not an ordinal or a sequence, except for indices. The smallest ordinal is $\emptyset$, the empty set.

A function σ from α to A (i.e. $\sigma \in A^\alpha$) is called a sequence of length α over A. For sequences σ and τ of length α and γ, respectively, we have the following sequence of length $\alpha + \gamma$, said to be obtained by concatenation:

$$\sigma || \tau : v \mapsto \begin{cases} \sigma(v) & \text{if } v < \alpha, \\ \tau(\delta) & \text{if } v = \alpha + \delta < \alpha + \gamma. \end{cases}$$

Obviously, concatenation is an associative operation on the class of all ordinals. If $\sharp A = 1$ then concatenation is essentially ordinal addition.

Let β be an ordinal $\geqslant 1$ and let $A^{<\beta}$ be the set of all sequences over A of length $< \beta$. We try to define a semigroup operation on the set $A^{<\beta} \cup \{0\}$ which extends concatenation, i.e. the product of two sequences σ and τ of length $< \beta$ will be the sequence $\sigma || \tau$ provided that $\sigma || \tau$ has length $< \beta$ (we also write $\sigma\tau$ or $\sigma \cdot \tau$ in this case). Any such semigroup will be called a concatenation monoid over A of bound β.

A set $A^{<\beta}$ with $A \neq \emptyset$ is closed under concatenation if and only if $\beta = \omega^\delta$ for some ordinal δ (because ordinal addition is closed within an ordinal β iff β has this form—as mentioned before Proposition 4.9). In this case there is just one concatenation monoid on $A^{<\beta} \cup \{0\}$ and this semigroup has no zero divisors. For $\delta = 1$ we have $A^{<\omega} = A^*$. If β is not of this form, say

$$\beta = \omega^{\delta_0} n_0 + \omega^{\delta_1} n_1 + \cdots + \omega^{\delta_k} n_k$$

in Cantor normal form (i.e. $\delta_0 > \delta_1 > \cdots > \delta_k$, $n_i \in \omega \backslash \{\emptyset\}$ for $i = 0, 1, \ldots, k$), with $n_0 > 1$ or $k \geqslant 1$, then there are generally several suitable operations (at least two in the most restricted case). Indeed, we may define

$$\sigma' \sigma'' \cdot \tau' \tau'' = \sigma' \, || \, \tau''$$

for sequences $\sigma', \sigma'', \tau', \tau''$ of length

$$l(\sigma') = \omega^{\delta_0} n_0 + \omega^{\delta_1} n_1 + \cdots + \omega^{\delta_k} (n_k - 1),$$

$$l(\sigma'') < \omega^{\delta_k},$$

$$l(\tau') = \omega^{\gamma_0} m_0 + \cdots + \omega^{\gamma_h} m_h < \beta, \quad \text{with } \gamma_h \geqslant \delta_k, \quad \text{and}$$

$$l(\tau'') < \omega^{\delta_k},$$

or

$$\sigma'\sigma'' \cdot \tau'\tau'' = 0$$

for all such sequences. Observe that in the semigroup $A^{<\omega^{\delta_0+1}}$ the concatenation of sequences of the form $\sigma'\sigma''$ and $\tau'\tau''$ can be performed and yields a sequence of length $\geqslant \beta$ whereas the concatenation of all other pairs of sequences of lengths $<\beta$ yields sequences of length $<\beta$. It follows that the semigroup on $A^{<\beta} \cup \{0\}$ with the latter multiplication is the Rees factor semigroup of the unique semigroup on $A^{<\omega^{\delta_0+1}}$ by the ideal of all sequences of length at least β. While all of the foregoing holds for an arbitrary ordinal $\beta \geqslant 1$ we now suppose that β is infinite.

Remark 5.1. Let $S^{\underline{1}} = A^{<\beta} \cup \{0\}$ be a concatenation monoid of bound

$$\beta = \omega^{\delta_0} n_0 + \omega^{\delta_1} n_1 + \cdots + \omega^{\delta_k} n_k \text{ with } \delta_0 \geqslant 1 \text{ (Cantor normal form).}$$

(1) For every $\sigma \in A^{<\beta}$ of length ω^{δ_0} and every $\tau \in A^{<\beta}$,
$\sigma\tau \neq 0$ implies $\sigma\tau = \sigma \,||\, \theta$ for some θ of length $<\omega^{\delta_0}(n_0-1)+\omega^{\delta_1}n_1+\cdots+\omega^{\delta_k}n_k$.
(2) S is right free.

Proof. (1) Assume that σ is not a prefix of $\sigma\tau$ and let γ be the smallest ordinal $<\omega^{\delta_0}$ such that either $\sigma\tau$ has length γ (hence smaller than the length of σ) or $(\sigma\tau)(\gamma) \neq \sigma(\gamma)$ (i.e. γ is the first index where $\sigma\tau$ and σ are different). Put $\sigma = \sigma'\sigma''$ with σ' of length $\gamma+1$. Then σ'' has length ω^{δ_0} and $\sigma''\tau = \xi$ is an element of $A^{<\beta}$, hence $\sigma\tau = \sigma' \,||\, \xi$ so that $\sigma\tau$ has length $\geqslant \gamma+1$ and $(\sigma\tau)(\gamma) = \sigma'(\gamma) = \sigma(\gamma)$, contradicting the assumption.
(2) The ideal $I = \bigcap_{1 \leqslant i < \infty} S^i$ consists of 0 and of all sequences of length at least ω and $S \backslash I = A^+$. In the case $I \neq \{0\}$ (i.e. $\beta > \omega$) the properties (a)–(c) of Section 4 are obvious. $\square$

A general classification of concatenation monoids has not yet been obtained. We just treat the case $\beta = \omega + 1$.

Let D be a concatenation monoid over A of bound $\omega+1$, i.e. $D = A^* \cup A^\omega \cup \{0\}$. If $\sigma \in A^\omega$ then $\sigma x = 0$ or $\sigma x = \sigma$ for every $x \in D$ (5.1). As a consequence, the ideal $B = A^\omega \cup \{0\}$ consists of idempotents and of elements of square 0. Moreover, the set $N = \{x \in B: x^2 = 0\}$ is an ideal of D, which is null, and B/N is a 0-disjoint union of minimal ideals of B/N whose $\mathscr{J}$-classes are left zero semigroups.

As in Section 4, let $\overline{A^\omega}$ denote the set of maximal indecomposable components of the left A^*-set A^ω and $\bar{\sigma}$ the component containing σ ($\sigma \in A^\omega$) (i.e. $\bar{\sigma} = \bar{\tau} \in \overline{A^\omega}$ iff $\sigma, \tau \in A^*\zeta$ for some $\zeta \in A^\omega$). For $\Sigma \in \overline{A^\omega}$ we define $A_\Sigma = \{a \in A: \sigma \notin (A\backslash\{a\})^\omega \text{ for all } \sigma \in \Sigma\}$ (this is the set of letters that appear infinitely often in any sequence $\sigma \in \Sigma$); the mapping $\Sigma \mapsto A_\Sigma$ does not depend on the specific multiplication of D.

Put

$$\mathscr{N}_D = \{\Sigma \in \overline{A^\omega}: \sigma \in N \text{ for all } \sigma \in A^\omega\}$$

and

$$\alpha_D(\Sigma)=\{a\in A\colon\ \sigma a\neq 0 \text{ for some } \sigma\in\Sigma\}.$$

If $a\in\alpha_D(\Sigma)$ then $\sigma a\neq 0$ for all $\sigma\in\Sigma$. If $\sigma\notin N$ and $\sigma\in\Sigma$ (hence $\Sigma\notin\mathcal{N}_D$) then $a\sigma\notin N$ if and only if $a\in\alpha_D(\Sigma)$. It follows that $A_\Sigma\subseteq\alpha_D(\Sigma)$ for all $\Sigma\notin\mathcal{N}_D$. If $\sigma\tau\neq 0$ ($\sigma,\tau\in A^\omega$) then $\tau\in(\alpha_D(\Sigma))^\omega$. Hence $\tau\notin(\alpha_D(\Sigma))^\omega$ implies $\sigma\tau=0$ for all $\sigma\in\Sigma$, and $\sigma\notin(\alpha_D(\Sigma))^\omega$ for all $\sigma\in\Sigma$ implies $\Sigma\in\mathcal{N}_D$ (this cannot happen if A is finite because then there exists a $\sigma\in(A_\Sigma)^\omega$). Finally, put

$$\eta_D=\{(\Sigma',\Sigma'')\in\overline{A^\omega}\times\overline{A^\omega}\colon\ \Sigma'=\Sigma'' \text{ or there exists a } \Sigma\in\overline{A^\omega}$$
$$\text{such that } \Sigma'\Sigma\neq\{0\},\Sigma''\Sigma\neq\{0\}\}.$$

This is an equivalence relation on $\overline{A^\omega}$ (because $\sigma\tau\neq 0,\sigma\rho\neq 0\Rightarrow\tau\rho\neq 0$) and $\eta_D\subseteq\alpha_D\alpha_D^{-1}$, i.e. $(\Sigma',\Sigma'')\in\eta_D$ implies $\alpha_D(\Sigma')=\alpha_D(\Sigma'')$ (because $\sigma\tau\neq 0\Rightarrow(\sigma a\neq 0\Leftrightarrow\tau a\neq 0)$. Moreover, $\Sigma\eta_D\subseteq\mathcal{N}_D$ implies $\Sigma\eta_D=\{\Sigma\}$, i.e. any η_D-class which is contained in $\mathcal{N}_D$ is trivial (because $\Sigma'\Sigma\neq\{0\}$ implies $\Sigma^2\neq\{0\}$, hence $\Sigma\notin\mathcal{N}_D$). Now we have $\sigma a=\sigma$ iff $a\in\alpha_D(\bar\sigma)$ ($\sigma\in A^\omega,a\in A$), and $\sigma\tau=\sigma$ iff $(\bar\sigma,\bar\tau)\in\eta_D$, $\bar\tau\notin\mathcal{N}_D$ and $\tau\in\alpha_D(\Sigma))^\omega$, i.e. D is completely determined by $\alpha_D,\mathcal{N}_D$, and η_D.

Conversely, consider a mapping

$$\alpha\colon\overline{A^\omega}\to\mathcal{P}(A)\quad\text{(the powerset of } A\text{)},$$

a subset

$$\mathcal{N}\subseteq\overline{A^\omega},$$

and an equivalence relation

$$\eta \text{ on } \overline{A^\omega}$$

such that

$$\alpha(\Sigma)\supseteq A_\Sigma\quad\text{for all } \Sigma\in\overline{A^\omega}\backslash\mathcal{N},$$
$$\eta\subseteq\alpha\alpha^{-1},$$
$$\Sigma\in\mathcal{N}\quad\text{if } \sigma\notin(\alpha_D(\Sigma))^\omega\quad\text{for all } \sigma\in\Sigma,$$

and

$$\eta\Sigma=\{\Sigma\}\quad\text{if } \eta\Sigma\subseteq\mathcal{N},$$

and define

$$\sigma v=\begin{cases}\sigma & \text{if } v\in(\alpha(\bar\sigma))^+,\\ 0 & \text{otherwise}\end{cases}\qquad(\sigma\in A^\omega,\ v\in A^+)$$

and

$$\sigma\tau=\begin{cases}\sigma & \text{if } (\bar\sigma,\bar\tau)\in\eta,\ \bar\tau\notin\mathcal{N},\text{ and } \tau\in(\alpha(\bar\sigma))^\omega,\\ 0 & \text{otherwise}\end{cases}\qquad(\sigma,\tau\in A^\omega),$$

all other non-zero products being given by concatenation. Then we have a concatenation monoid which we denote by $D(\alpha, \mathcal{N}, \eta)$. Clearly, $\alpha_{D(\alpha,\mathcal{N},\eta)} = \alpha$, $\mathcal{N}_{D(\alpha,\mathcal{N},\eta)} = \mathcal{N}$, and $\eta_{D(\alpha,\mathcal{N},\eta)} = \eta$, whereas $D = D(\alpha_D, \mathcal{N}_D, \eta_D)$ for every concatenation monoid D of bound $\omega + 1$.

Observe that in a concatenation monoid $D = D(\alpha, \mathcal{N}, \eta)$, putting $S = D \backslash \{1\}$ and $I = \bigcap_{1 \leqslant i < \infty} S^i = A^\omega$, we have

$$\{\sigma\colon \bar{\sigma} \in \mathcal{N}\} \cup \{0\} = \{x \in I\colon Ix = \{0\}\}$$

and

$$\{\sigma\colon \alpha(\bar{\sigma}) = \emptyset\} \cup \{0\} = \{x\colon xS = \{0\}\}$$

If U is a proper ideal of D such that $a^{-1}U = U$ for all $a \in A$ (this implies $U \subseteq I$, hence Remark 4.5 applies) then

$$U \backslash \{0\} = \bigcup_{\Sigma \in \mathcal{U}} \Sigma$$

for some subset $\mathcal{U}$ of $\overline{A^\omega}$. We may modify α, $\mathcal{N}$, and η in such a way that

$$\alpha(\Sigma) = \emptyset, \;\; \Sigma \in \mathcal{N} \quad \text{and} \quad \eta\Sigma = \{\Sigma\} \;\text{ for all } \Sigma \in \mathcal{U},$$

without changing D/U. It follows that every right free Rees factor of a concatenation monoid of bound $\omega + 1$, by an ideal contained in I (4.5), is uniquely determined by a quadruple $(\alpha, \mathcal{N}, \eta, \mathcal{U})$ with $(\alpha, \mathcal{N}, \eta)$ as before and $\mathcal{U} \subseteq \{\Sigma \in \overline{A^\omega}\colon \alpha(\Sigma) = \emptyset\}$. We write $D(\alpha, \mathcal{N}, \eta, \mathcal{U})$ for the monoid determined by these data.

A complete classification of the right free semigroups with 0 that correspond to Rees factors of concatenation monoids over A of bound $\omega + 1$ would require to determine a representative quadruple for every isomorphism class. It is possible that, with some set-theoretic effort, such a characteristic quadruple can be found. Observe that there is precisely one quadruple with $\mathcal{U} = \overline{A^\omega}$ (with $\alpha(\Sigma) = \emptyset$, $\mathcal{N} = \overline{A^\omega}$, $\eta = \Delta_{\overline{A^\omega}}$), characteristic for the free monoid with 0 over A, and precisely one quadruple with $\mathcal{N} = \emptyset$ and $\eta = \overline{A^\omega} \times \overline{A^\omega}$ (with $\alpha(\Sigma) = A, \mathcal{U} = \emptyset$), characteristic for the concatenation monoid without zero divisors.

We end this section with a structural characterization of concatenation monoids of bound $\omega + 1$ and of their right free Rees factor semigroups.

Theorem 5.2. *Suppose $S \neq \{0\}$. The following are equivalent*:

(1) *S is right free, $\mathcal{R} = \Delta_S$, and every maximal chain of non-zero principal right ideals has order type $\omega + 1$ and is uniquely determined by any infinite subchain.*

(2) *S is right free and the intersection of any infinite chain of non-zero principal right ideals contains precisely one non-zero element.*

(3) *$S^{\underline{1}}$ is isomorphic with a concatenation monoid of bound $\omega + 1$.*

Proof. It is easily checked that (3) implies (1). Assume (1) and let $\mathcal{C}$ be an infinite chain of non-zero principal right ideals. $\mathcal{C}$ is contained in a maximal chain $\mathcal{G}$ by Zorn's lemma (this appears to be the first point where we employ the axiom of choice). Let R be the last member of $\mathcal{G}$. If R belongs to $\mathcal{C}$ then R is the intersection of $\mathcal{C}$. If $R \notin \mathcal{C}$

then $\mathscr{C}$ is cofinal with $\mathscr{G}\backslash\{R\}$, hence both chains have the same intersection. But R is the only non-zero principal right ideal contained in this intersection (by the uniqueness property of $\mathscr{G}$) which implies that the intersection of $\mathscr{C}$ is R. $R\backslash\{0\}$ is an $\mathscr{R}$-class, hence contains a single non-zero element by assumption. Thus (1) implies (2).

Now let (2) be satisfied. We may assume $S = A^+ \cup \bigcup_{\sigma\in A^\omega} B_\sigma$ where B_σ is the intersection of the principal right ideals $vS^{\underline{1}}$, $\sigma \in vA^\omega$ (Section 4). For every $\sigma \in A^\omega$ there is precisely one non-zero element contained in B_σ, hence there is an isomorphism to a concatenation monoid on $A^* \cup A^\omega \cup \{0\}$. $\square$

If S is right free we can fill up S by adding a single non-zero element s_σ to any B_σ such that $B_\sigma = \{0\}$, e.g. we may define a multiplication on $\tilde{S} = A^+ \cup B \cup U$ with $U = \{s_\sigma : B_\sigma = \{0\}\}$ in such a way that $(B \cup U)U = U\tilde{S} = \{0\}$. We always choose $vs_\sigma = s_{v\sigma}$ $(v \in A^+)$. Then U is an ideal of $\tilde{S}$ satisfying $a^{-1}U = U$ for every $a \in A$ (hence contained in $\tilde{I} = \bigcap_{1\leqslant i<\infty} \tilde{S}^i$), $\tilde{S}$ is right free (4.5), and $S = \tilde{S}/U$.

Corollary 5.3. *Suppose* $S \neq \{0\}$. *The following are equivalent*:

(1′) *S is right free, $\mathscr{R} = \Delta_S$, and every maximal chain of non-zero principal right ideals has order type $\leqslant \omega + 1$ and is uniquely determined by any infinite subchain.*

(2′) *S is right free and the intersection of any infinite set of principal right ideals contains at most one non-zero element.*

(3′) *$S^{\underline{1}}$ is isomorphic with a Rees factor semigroup of a concatenation monoid of bound $\omega + 1$ over A by an ideal U such that $a^{-1}U = U$ for all $a \in A$.*

6. Biprojective semigroups with 0

A left-right biobject over S consists of a set M, a nullary operation defining 0, and multiplications $S \times M \to M$ and $M \times S \to M$ such that M together with 0 and the former multiplication yields a left S-object, M together with 0 and the latter yields a right S-object, and $(s_1 m)s_2 = s_1(ms_2)$ for all $s_1, s_2 \in S$ and $m \in M$. The category of biobjects over S has the operation-preserving mappings as morphisms. ${}_SM_S$ indicates that M is considered as an object of this category. We have the obvious forgetful functors into the categories of left S-objects and of right S-objects. There is a functor from the product of these two categories into the category of biobjects which maps $({}_SL, N_S)$ to ${}_SL \otimes N_S$, the tensor product of L and N, considered as right and left unital object over the two-element monoid with 0 (i.e. ${}_SL \otimes N_S = L \times N/((L \times \{0\}) \cup \{0\} \times N))$). The restricted functor $- \otimes S^{\underline{1}}_S$ from the category of left S-objects to the category of left-right biobjects over S is left adjoint to the aforementioned forgetful functor (the corresponding holds for the functor ${}_SS^{\underline{1}} \otimes -$). This fact could be used to explain the special role of the biobject ${}_SS^{\underline{1}} \otimes S^{\underline{1}}_S$ in the following.

Clearly, S is a biobject over S. We will see that the property of being projective as a biproject, i.e. ${}_SS_S$ projective, imposes a much stronger restriction on S than being left and right projective. The definitions of projectivity and of freeness for biobjects over S are taken from category theory [12] and their basic properties are completely analogous to the case of right S-objects (cp. the fifth to ninth paragraph of Section 2). However, it should be emphasized that ${}_SS^{\underline{1}}_S$ is not a free bioperand. The place of $S^{\underline{1}}_S$ is taken by ${}_SS^{\underline{1}} \otimes S^{\underline{1}}_S$ (which is free over $\{1 \otimes 1\}$, where $1 \otimes 1 = (1,1)$ in the standard Rees factor object of ${}_SS^{\underline{1}} \times S^{\underline{1}}_S$).

An ${}_SM_S$ is free over X, where $X \subseteq M$, if and only if M is a 0-disjoint union of the biobjects $S^{\underline{1}}xS^{\underline{1}}$ $(x \in X)$, subobjects of ${}_SM_S$, and $s_1xu_1 = s_2xu_2$, $s_1, s_2, u_1, u_2 \in S^{\underline{1}} \backslash \{0\}$, implies $(s_1, u_1) = (s_2, u_2)$, i.e. $S^{\underline{1}}xS^{\underline{1}} \cong S^{\underline{1}} \otimes S^{\underline{1}}$.

A non-zero principal ideal of S, i.e. $S^{\underline{1}}xS^{\underline{1}}$ with $x \in S$, can never satisfy this condition (because $xx1 = 1xx$ and $1 \notin S$). Hence no non-zero ideal of S is ever free as a biobject. Especially, S is free as a biobject if and only if $S = \{0\}$.

A biobject ${}_SM_S$ is projective iff M is a retract of a free biobject, hence, as is easily seen, a 0-disjoint union (i.e. a coproduct) of biobjects of the form $S^{\underline{1}}exfS^{\underline{1}}$, where e and f are idempotents of $S^{\underline{1}}$ and the mapping $s_1e \otimes fs_2 \mapsto s_1exfs_2$ $(s_1e \otimes fs_2 \in S^{\underline{1}}e \otimes fS^{\underline{1}})$ is an isomorphism.

If $x \in S$ and if $s_1e \otimes fs_2 \mapsto s_1exfs_2$ is an isomorphism from $S^{\underline{1}}e \otimes fS^{\underline{1}}$ to $S^{\underline{1}}exfS^{\underline{1}}$ then e and f belong to S (because $e = 1$ and $xf \cdot x \cdot f1 = 1 \cdot x \cdot fxf$ would imply $1 = xf \in S$), and correspondingly for f), hence $S^{\underline{1}}exfS^{\underline{1}} = SexfS$.

Suppose that ${}_SS_S$ is projective. Then S is a 0-disjoint union of (0 or more) non-zero projective principal ideals. We may assume that S coincides with one of these ideals, say $S = SexfS$, with ${}_SS_S \cong S^{\underline{1}}e \otimes fS^{\underline{1}} = Se \otimes fS$ as above. This isomorphism implies that S is the 0-disjoint union of the right ideals $sexfS$, $se \neq 0$, and that all of these are isomorphic with fS. Correspondingly, S is the 0-disjoint union of the left ideals $Sexfs$, $fs \neq 0$, all of them isomorphic with Se. Moreover, e belongs to S, say $e = sexft$, hence $ex = sexftx$, which implies $se = e$ so that $e\mathscr{R}exf$. Correspondingly, $exf\mathscr{L}f$ which shows $e\mathscr{D}f$. It follows that fS and eS are isomorphic, hence S is a 0-disjoint union of right ideals isomorphic with eS and also a 0-disjoint union of left ideals isomorphic with Se, e an idempotent of S. It is known [9] that such semigroups are isomorphic with Rees matrix semigroups over eSe, $S \cong (eSe; p\colon \Lambda \times I \to eSe)$. In this case eSe is a two-element monoid because $e = exft$, hence $esexf = ese \cdot x \cdot f = e \cdot x \cdot ftsexf$, for some t, so that $ese = e$ or $ese = 0$. Therefore $S \cong (\{0,1\}; p)$, with $p(\Lambda \times I) = \{0,1\}$ so that $S^2 = S$ ($S \neq \{0\}$ implies $e \neq 0$, hence S cannot be null). We have proved the non-calculatory part of the following statement.

Theorem 6.1. *${}_SS_S$ is projective iff $S = S^2$ and S is isomorphic with a 0-disjoint union of ideals which are Rees matrix semigroups over the two-element monoid $\{0,1\}$.*

It is easily calculated that every Rees matrix semigroup $S = (D; p)$, D a monoid with 0, which is globally idempotent (i.e. $S^2 = S$), is a 0-disjoint union of left ideals, all of them isomorphic with an appropriately chosen idempotent-generated principal left ideal, and correspondingly for right ideals. Hence any 0-disjoint union of ideals which are semigroups of this type is left and right projective. Especially, the biprojective semigroups S are left and right projective. This may also be seen directly because ${}_SM_S$

projective implies ${}_SM$ and M_S projective. A biprojective semigroup S has the following additional properties:
(A) $xyx \neq 0$ implies $xyx = x$,
(B) $SeSfS = SfSeS$ for any idempotents e and f of S.

The latter property holds in any idempotent semigroup. However, an idempotent semigroup satisfies condition (A) if and only if it is a 0-disjoint union of 0-simple semigroups, i.e. of rectangular bands with 0 adjoined. Hence any idempotent semigroup satisfying (A) is biprojective (6.1).

There are idempotent semigroups that are left and right projective, but not biprojective, e.g. the 7-element semigroup $\langle e, f; e^2 = e, f^2 = f, efef = ef, fefe = fe\rangle \cup \{0\}$. On the other hand, there are semigroups satisfying (A) that are left and right projective, but not biprojective, e.g. the semigroup $\langle e, f, 0;\ e^2 = e, f^2 = f, efe = fef = 0e = e0 = 0f = f0 = 0\rangle$. We will see that (A) and (B) together suffice to characterize the biprojective among the right projective semigroups. We will also use the following variation of (A).

(A$'$) $exe \neq 0$ together with $e = e^2$ implies $exe = e$.

If S is a union of idempotent-generated ideals, $S = \bigcup_{e^2=e} SeS$, then (A) and (A$'$) are equivalent. Indeed, if $xyx \neq 0$, $x = uer$ with $e = e^2$ then $ueryuer \neq 0$, hence $eryue \neq 0$ so that $eryue = e$ by (A$'$). It follows that $xyx = x$.

In a semigroup S satisfying (A$'$) every non-zero idempotent is primitive, hence every principal factor $\mathscr{J}e \cup \{0\}$, e a non-zero idempotent, is completely 0-simple. Moreover, every subgroup of S is a one-element group. Another consequence of (A) is the following:

(A$''$) If a non-zero idempotent e is contained in a right ideal uS with $u \in S$ then $eS = uS$.

Indeed, $e = ux = uxux \neq 0$ implies $uxu \neq 0$, hence $eu = uxu = u$.

We also observe that by (A) we have $z^3 = 0$ or $z^3 = z$ for every $z \in S$, and that $z^3 = z$ implies $zz = zzzz = z$. Hence the following consequence of (A):

(A$'''$) Every $s \in S$ is an idempotent or satisfies $s^3 = 0$.

The following is an essential step in the proof of the subsequent theorem.

Remark 6.2. Assume that S satisfies (A) and (B), that $S = S^2$, and that S is a 0-disjoint union of principal right ideals. Then S is a 0-disjoint union of idempotent-generated principal ideals. Moreover, any non-zero idempotent generates a maximal principal ideal and a maximal principal right ideal.

Proof. Assume $S \neq \{0\}$ and let $u = xy$ be a generator of a maximal principal right ideal. Then $x \in uS^1$, hence $u = uz$ for some $z \in S$. Now $u = uzzz \neq 0$, hence $zzz \neq 0$ so that z is an idempotent. If $SeS \cap SfS \neq \{0\}$ with e and f idempotents, say $set = qfr \neq 0$ then set is contained in a maximal principal right ideal, say in uS with $u = uz, z = z^2$. Then q is contained in uzS, say $q = uzk$, hence $qfr = uzkfr \neq 0$. It follows that $zkf \neq 0$ and that $zkf = q_1fq_2zq_3 = zq_1fq_2zq_3f$ for some $q_1, q_2, q_3 \in S$ by (B). Now $zq_1fq_2z \neq 0$ and $fq_2zq_3f \neq 0$, hence $zq_1fq_2z = z$ and $fq_2zq_3f = f$ by (A), hence $SzS = SfS$.

Correspondingly, $SzS = SeS$. That e generates a maximal principal right ideal is just property (A$''$). □

Theorem 6.3. *Suppose $SeSfS = SfSeS$ for any idempotents e, f of S. Then the following are equivalent*:
(1) $S = S^2$, *S is right projective and satisfies* (*A*$'$).
(2) $S = S^2$, *S satisfies* (A), *S is a* 0-*disjoint union of principal right ideals and a* 0-*disjoint union of principal left ideals*, *and $ux \neq 0, xv \neq 0$ implies $uxv \neq 0$* $(u, x, v \in S)$.
(3) ${}_S S_S$ *is projective.*

Proof. We have observed that (3) implies (1). Assume (1). Then S is a 0-disjoint union of principal right ideals and $S = \bigcup_{e^2=e} SeS$ (3.5), hence S satisfies (A) (as observed above). In order to see that (2) holds we will first show that S is a 0-disjoint union of principal left ideals. Consider a maximal principal right ideal $u'S$ containing a non-zero idempotent e. Then $u'S = eS$ by (A$''$). We also know that SeS is 0-disjoint from any other idempotent-generated principal ideal (6.2). Assume $Sex \cap Sey \neq \{0\}$. We will see that $ex = ey$, hence Sex is a maximal principal left ideal and SeS is the 0-disjoint union of the right ideals Sex with $ex \neq 0$.

Assume $sex = tey \neq 0$. There exists a $u \in S$ and an idempotent e_u such that uS is a maximal principal right ideal, $se, te \in uS$, $u = ue_u$, and $e_u x \mapsto ux$ is an isomorphism from $e_u S$ to uS. Then $se = ue_u qe, te = ue_u re$, say, so that $e_u qe \neq 0, e_u re \neq 0$. The idempotents e_u and e generate the same ideal (6.2), and therefore belong to the same completely 0-simple principal factor. Thus there exist d and g such that $(e_u, de_u) \in \mathscr{L}$, $de_u = eg$, $(eg, e) \in \mathscr{R}$ and we have $de_u qeg = de_u reg = eg$ by (A). It follows that $e_u qe = e_u re$, hence $ue_u qex = sex = tey = ue_u qey$, so that $e_u qex = e_u qey$, hence $eggex = de_u qex = de_u qey = eggey$. Since $egqe = e$ by (A$'$) we have $ex = ey$ as desired.

Now suppose $ux \neq 0$ and $xv \neq 0$. Then $u, x, v \in SeS$ for some idempotent e. SeS is a 0-disjoint union of principal right ideals, each of them isomorphic with some idempotent-generated principal right ideal contained in SeS. As all non-zero idempotents of SeS belong to the same completely 0-simple principal factor, hence to the same $\mathscr{D}$-class, each maximal principal right ideal is isomorphic with eS, hence has the form $u_i S$ with $u_i e = u_i$ and $es \mapsto u_i s$ an isomorphism from eS to $u_i S$. Now $S = \coprod_{i \in I} u_i S$ with every u_i as described (the set$\{u_i$: $i \in I\}$ is uniquely determined in our case, but this is not needed at this point), hence $u = u_i es, x = rez, v = u_j et$ for some $i, j \in I$ and $s, r, z, t \in S$. It follows that $u_i esre = u_i e \neq 0$ and $ezu_j et = et \neq 0$ by (A), hence $uxv = u_i et$ which is non-zero because $es \mapsto u_i es$ is an isomorphism.

Now assume (2). By (6.2) we know that S is a 0-disjoint union of principal ideals SeS with $e = e^2$. Moreover, $eSe = \{e, 0\}$ by (A). In order to reach (3) it suffices to show that SeS is a 0-disjoint union of left ideals isomorphic with Se and a 0-disjoint union of right ideals isomorphic with eS (as in the proof of (6.1)).

For this consider an arbitrary s with $se \neq 0$ and the right ideal seS. This right ideal is contained in a maximal principal right ideal of the form uS (because $S = S^2$). If $u = xey$ then $se = xeyr = xeyre$ for some r, hence $se = xe$ by (A), so that $uS = seS$. By the same argument we have that $seS = s'eS \neq \{0\}$ implies $se = s'e$. Correspondingly, $Set \neq \{0\}$ is a maximal principal left ideal and $Set = Set' \neq \{0\}$ implies $et = et'$. Remember that different maximal principal left ideals are 0-disjoint. Now the mapping $et \mapsto set$ is an isomorphism from eS to seS because $se \neq 0, et \neq 0$ implies $set \neq 0$ and $set = set' \neq 0$ implies $Set = Set'$, hence $et = et'$. Correspondingly for the maximal principal left ideals. $\square$

A semigroup containing a subsemigroup which is free cannot satisfy (A). Hence $S \backslash S^2 = \emptyset$ (and therefore $S = S^2$) in any right projective S satisfying (A).

Corollary 6.4. *${}_S S_S$ is projective iff S is right projective and satisfies* (A) *and* (B).

7. Generalizations

We will briefly consider some directions in which the preceding study may be generalized: monoids, conditions weaker than projectivity, variations of the variety of objects, and enriched semigroups such as rings. We may come back to the first and last of these topics in the future. All proofs are omitted. Given a semigroup S with 0, let P, T, B have the same meaning as in Section 3. A and I are used somewhat more generously in Section 7.1, but have the former meaning in all of the rest.

7.1. Projective ideals in monoids

As mentioned in Section 2, the category of unital right $S^{\underline{1}}$-objects is isomorphic with the category of right S-objects (by the forgetful functor). Clearly, a monoid D is isomorphic with a monoid $S^{\underline{1}}$ if and only if the $\mathscr{J}$-class of 1 is trivial. Therefore, the foregoing sections have dealt with monoids D such that the greatest proper ideal of D, say V, satisfies $D/V = \{1, 0\}$ and is projective as a unital right D-object. In a monoid of this sort V is also the greatest proper right ideal. We may try to weaken the condition that $D/V = \{1, 0\}$, e.g. we may ask for the structure of monoids whose greatest proper right ideal or greatest proper two-sided ideal is projective as a right D-object.

First let V be an arbitrary projective right ideal of a monoid D with 0. There exist a subset A and a subset U of D such that $V = \coprod_{z \in A \cup U} zD$, with $d \mapsto zd$ an isomorphism from D_D to zD for every $z \in A$, and $e_z d \mapsto zd$ an isomorphism from $e_z D$ to zD, e_z some idempotent not contained in the $\mathscr{D}$-class of 1, for every $z \in U$. A and U are not uniquely determined by V in general: Every $a \in A$ may be replaced by an $a' \in a \cdot (\mathscr{H} 1)$ and every $u \in U$ may be replaced by a $u' \in u \cdot (\mathscr{H} e_u)$ (and a $\tilde{u}$ appropriate for any other idempotent $\tilde{e} \in \mathscr{D} e_u$ could have been chosen in the first place). Now

$$D = A^* C \cup \left(\bigcap_{1 \leqslant i < \infty} A^i V \right) \cup A^* U D,$$

where $C = D \setminus V$, and A^* stands for the free monoid over A (up to isomorphism). The right ideals $\bigcap_{1 \leqslant i < \infty} A^i V$ and A^*UD are 0-disjoint and their union is disjoint from the set A^*C. Moreover, A^*C is an unambiguous product of A^* and C and A^*UD is an unambiguous product of A^* and UD. If all e_u $(u \in U)$ can be chosen in V, which is the case if V is the greatest proper right ideal of D, then

$$V = A^+C \cup I \text{ (a disjoint union)},$$

where $I = (\bigcap_{1 \leqslant i < \infty} A^i V) \cup A^*UV$. We do not know whether $A^+C \cup \{0\}$ is always a subsemigroup or I an ideal of D.

If C is a group (hence the group of units) then A^*C is a subsemigroup (and A^+C as well). The simplest example is the free product of a free monoid and a group (qua monoids), with an adjoined zero.

Observe that $D = K^{\underline{0}}$, with K the bicyclic semigroup, is a monoid with 0 in which every right ideal is free, hence projective, e.g. we have $D = \{y^2\}^* \cdot C \cup \{0\}$ with $C = \{1, x, x^2, \ldots, y, yx, yx^2, \ldots\}$.

Not much new arises for two-sided ideals V with respect to biprojectivity: Assuming $V \neq \{0\}$ we have that ${}_DV_D$ is projective and non-null iff ${}_VV_V$ is projective.

7.2. Flatness properties

Various generalizations of projectivity have been considered for rings and other associative structures. For semigroups, flatness and interpolation properties have been in the foreground, primarily in the context of monoids K and non-centered unital K-sets (K-acts). The strongest property studied so far is pullback-flatness (i.e. the property of a right K-set M that the functor $M \otimes_K -$ from left K-sets to sets preserves pullback diagrams).

As shown by Bulman–Fleming [4], this is equivalent to the following interpolation condition on M:

(PF) If $ms = m's'$ and $mt = m't'$ $(m, m' \in M, s, s' \in K)$

then $m = m''q,\ m' = m''r,\ qs = rs',\ qt = qt'$ for some $m'' \in M$ and $q, r \in K$.

Moreover [4], pullback-flatness implies equalizer flatness so that pullback-flat K-sets coincide with a type of K-set considered earlier by Stenström [18], who showed that these K-sets are precisely the directed colimits of finitely generated free K-sets.

If in (PF) we replace K by $S^{\underline{1}}$ (S a semigroup with 0) we obtain a condition that characterizes pullback-flatness for S-operands, even after restriction to centered S-operands, i.e. S-objects. However, pullback-flatness for S-objects is not implied by projectivity. Indeed, it is easily seen that an S-object M satisfies (PF) if and only if

(LC) M is locally cyclic (i.e. any finite subset is contained in a principal subobject) and

(E) $ms = ms'$ $(m \in M, s, s' \in S^{\underline{1}})$

implies $m''q = m,\ qs = qs'$ for some $m'' \in M$ and $q \in S^{\underline{1}}$.

Clearly, any locally cyclic S-object is indecomposable. Therefore pullback-flatness for S_S just generalizes the right indecomposable right projective semigroups considered in Corollary 4.2 (of unknown structure themselves). Observe though that all S with S_S locally cyclic which are (qua semigroups with 0) directed colimits of free semigroups with 0 do satisfy (PF), among them the rational interval $[0,1)$, with ordinary multiplication (the real interval $[0,1)$ is also pullback-flat).

A weaker condition, more adapted to semigroups with 0, and weaker then (PF) restricted to maximal indecomposable subobjects, is the following:

(PF_0) If $ms = m's' \neq 0$ and $mt = m't' \neq 0$ $(m, m' \in M, s, s' \in S^{\underline{1}})$

then $m = m''q,\ m' = m''r,\ qs = rs',\ qt = rt'$

for some $m'' \in M$ and $q, r \in S^{\underline{1}}$.

Clearly, (PF_0) is inherited by Rees factor objects. Unfortunately, an interpretation of (PF_0) as a limit-preservation property has not yet been found. However, (PF_0) implies plain flatness, i.e. the property that $M \otimes_s -$ preserves monomorphisms [5,12]. (PF_0) is equivalent to

(LC_0) every maximal indecomposable of M is locally cyclic,

combined with

(E_0) $ms = ms' \neq 0$ $(m \in M, s, s' \in S^{\underline{1}})$

implies $m''q = m,\ qs = qs'$ for some $m'' \in M$ and $q \in S^{\underline{1}}$.

Any Rees factor semigroup of a right projective semigroup satisfies (PF_0). The converse does not hold as the aforementioned examples of pullback-flat semigroups show.

But assume that S is a union of maximal principal right ideals. Then, as an analogue to Theorem 3.5, we have that S satisfies (PF_0) iff the following properties hold: (a) $A = S \backslash S^2$ is a set of left 0-cancellable elements, (b) and (c) as before, and (d) for every $u \in S$ such that uS is a maximal principal right ideal, $us = ut$ implies $hs = ht$ for some $h \in H_u = \{h \in S^{\underline{1}} : uh = u\}$. Especially, the subsemigroup H_u is left collapsible (i.e. for any h, h' there exists h'' with $h''h = h''h'$ [11,12]).

Property (d) allows to decompose an S satisfying (PF_0) 0-disjointly into a subsemigroup $S' = T \cup B \cup (TP \cap S') \cup (P \cap S')$ and a right ideal S'' satisfying $(S'')^2 = S''$, both satisfying (PF_0) (as right objects over themselves). Nothing is known about S'' in general, but S' can again be derived from $T \cup B$.

It can be shown that the right ideal $T \cup B$, in an arbitrary semigroup S satisfying (PF_0) (i.e. non-principal maximal indecomposable right ideals are admitted), is the largest subsemigroup S_1 of S satisfying $S_1 \backslash S_1^2 \subseteq T$, $S_1 = (S_1 \backslash S_1^2) S_1^{\underline{1}}$, and ($\mathrm{PF}_0$). Moreover, $T \cup B$ is a Rees factor semigroup of a right free semigroup $\tilde{S}$ with $\tilde{S} \backslash \tilde{S}^2 = A$. If A is a set of left cancellable elements then $T \cup B$ is right free.

The imposition of modest finiteness conditions brings (PF_0)-semigroups close to right projective semigroups, even the S''-part. E.g. M_L and M^R (the minimal and the maximal condition on principal left and principal right ideals, respectively) together with (PF_0) yield $B=\{0\}$, hence $S'=T$; moreover, S'' is a 0-disjoint union of principal right ideals isomorphic with Rees factors of idempotent-generated principal right ideals. It can be expected that these S are again Rees factor semigroups of right projective semigroups.

7.3. Subvarieties

It can be shown that a free idempotent semigroup with 0 (i.e. a semigroup $S=I^{\underline{0}}$ where I is a free band) is right projective as a semigroup with 0, but not right free. A free semigroup from the larger class of semigroups with 0 given by the equation $xy^2=xy$ is not even right projective. Similarly, a nil semigroup is never right projective. A commutative semigroup S with 0 which is right free is an ideal extension of a semigroup by an infinite cyclic semigroup as considered in Proposition 4.1. A commutative right projective semigroup with 0 is a 0-disjoint union of ideals, at most one a right free semigroup and the others generated by idempotents (i.e. commutative semigroups having an identity element).

These examples suggest a modification of the setting adopted so far. Clearly, the notion of projectivity depends on the considered category (as already observed in Section 2). We therefore restrict the class of relevant objects by imposing a set of semigroup-with-0-object identities, i.e. of identities of the form $zv=zw$, where z is an indeterminate that refers to an element of the S-object and v,w are members of the free monoid with 0 over indeterminates $x,y,\dots$ that refer to elements of S.

For example, consider the single identity $zx_1x_2\dots x_n=0$ (we write 0 instead of $z0$). It turns out that S_S is right projective as an S-object satisfying this identity iff S is free as a nilpotent semigroup of index $n+1$. In comparison, there is a wider class of right projective semigroups with respect to the identity $zx^n=0$ (where $n\geqslant 2$). In this context, a finite semigroup is right projective iff it is free in the equational class determined by $yx^n=0$, but infinite right projective semigroups need not be free in this equational class. Since all of these semigroups are nil so that non-zero idempotents do not exist there can never be a proper projective (i.e. non-right free) component.

In contrast, consider right projective semigroups with 0 with respect to the set $\{zx^2=zx, zxy=zyx\}$ of identities. Every semilattice with 0 having an identity element is right projective, as are 0-disjoint unions of such semigroups, and there are various versions of right free semigroups. Generally, with respect to a set $\{zv=zw\colon (v,w)\in\Sigma\}$ of object identities, any semigroup that is free in the semigroup-with-0 variety given by $\{tv=tw\colon (v,w)\in\Sigma\}$ (t an element not appearing in any v or w), is right free, and an $S=S^2$ is right projective iff S right projective in the unrestricted sense and belongs to the said semigroup-with-0 variety.

The foregoing equational approach may be taken as an indication that it is not a priori advisable to consider monoids instead of semigroups. However, it is also not generally advisable to consider semigroups with 0 instead of arbitrary semigroups (e.g. if one wishes to investigate semigroups in the class given by $zxy^2=zx$).

It remains to discuss the case of commutative semigroups. For this we consider $S-S$-biobjects on which we impose the identity $xz=zx$. It turns out that the biprojective semigroups belonging to this class have a somewhat analoguous structure to right projective semigroups in general. Especially, every free commutative semigroup with 0 is biprojective. There is an interesting class of monoids arising from bi-free commutative semigroups with 0 that is analoguous to the concatenation monoids considered in Section 5 (monoids of multisets with multiplicities taken from the set of cardinals below some fixed κ, with 0 adjoined).

7.4. Rings

Up to this point we have considered semigroups whose underlying categorical objects were pointed sets. The same questions as before may be explored for semigroups over richer objects (topological spaces, ordered sets, algebraic systems). For the special case of rings (qua semigroups over abelian groups) we list the following observations.

Every semigroup ring $R=\mathbf{Z}S$ of a right projective semigroup S with 0 is right projective as a right R-module. This includes the free rings and the rings of $I\times I$-matrices over $\mathbf{Z}$ (I any index set) having finitely many non-zero entries. The corresponding rings of matrices over any ring D with identity element are right projective as well. More generally, there are Rees matrix rings (of rectangular matrices over D, defined via a sandwich matrix p) that are right projective but not semigroup rings over any ring. Especially, every semiprime ring with minimal condition on principal left (or right) ideals (cf. [2]) is left and right projective. It remains to be seen whether all of these and possible other rings are biprojective.

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